Modelisation and Simulation of Assemblies of Structures

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Lecture Notes

Abstract

Lecture Notes of the course on modelisation and simulation of assemblies of structures in the second year of Master TACS and DSME. It Deals with boundary conditions, assemblies, contact ans friction. Also focusses on non matching meshes.

Contents

Chapter 1

Introduction

1.1 Why modeling assemblies

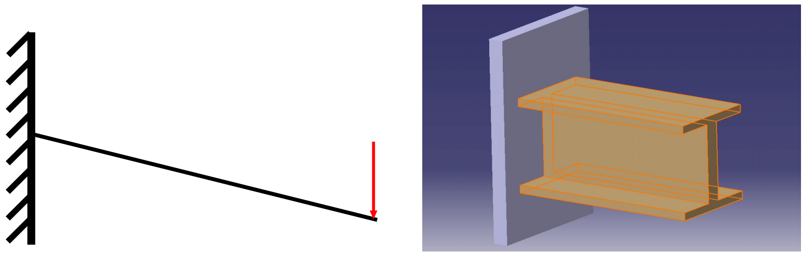

Figure 1.1: Modeling a cantilever : beam contion (left) and 3D condition (right)

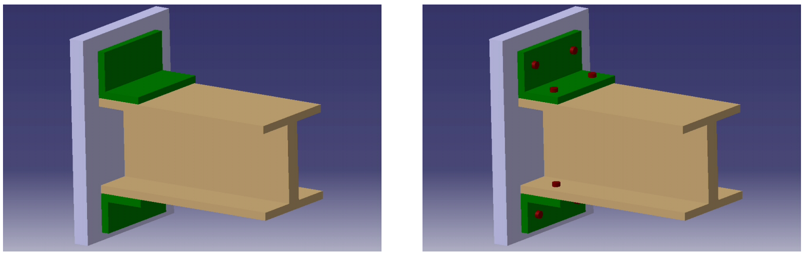

Figure 1.2: Modeling a cantilever : more complex 3D models

1.2 Contents

1.3 Notations and recalls

1.3.1 Local problem

On a omain Ω submitted to body forces $\underline{f}^d$, to surface forces $\underline{F}^d$ on the part of its boundary $\partial\Omega_F$ and to prescribed displacements $\underline{u}^d$ on the part of the boundary $\partial\Omega_u$, the displacement field $\underline{u}$ and the stress field $\underline{\underline{\sigma}}$ are solutions of the following local problem:

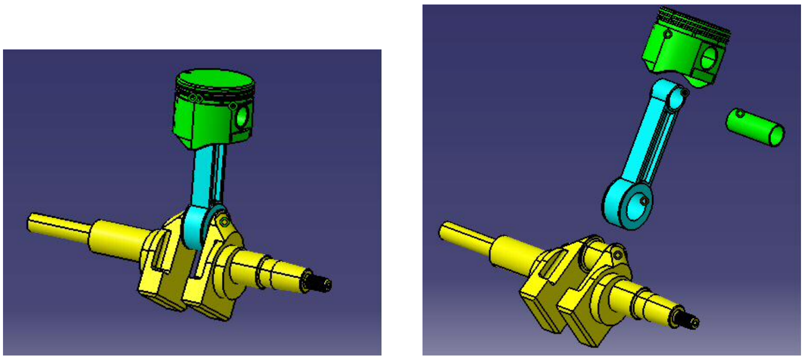

Figure 1.3: Modeling of an assembly

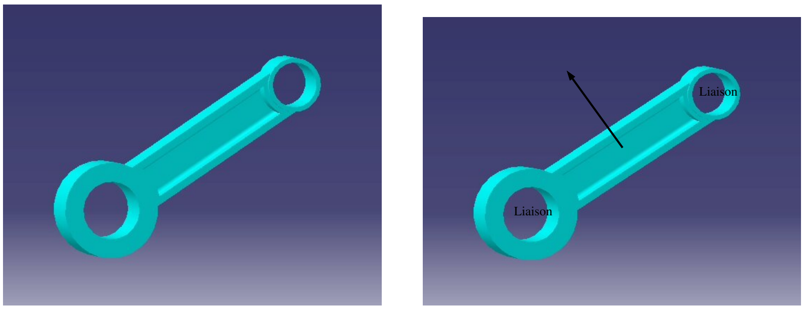

Figure 1.4: Simple modeling of the connecting rod

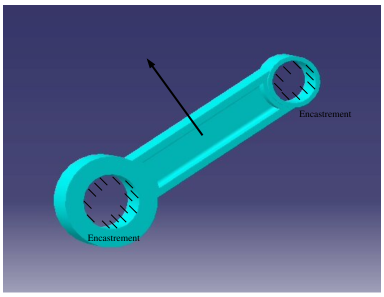

Figure 1.5: Modeling cantilever connections

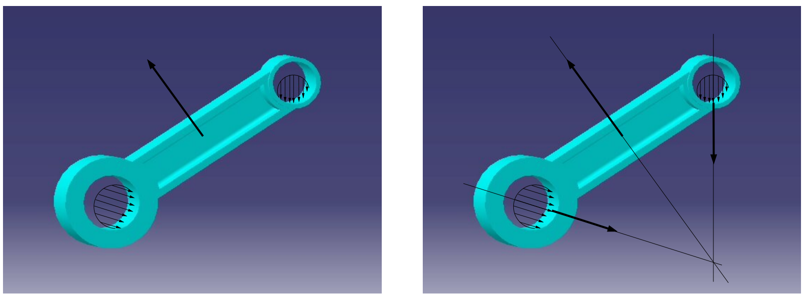

Figure 1.6: Modeling forces conditions on the connection

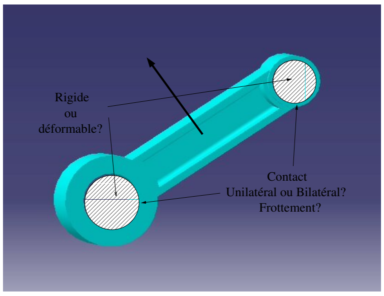

Figure 1.7: Modeling cantilever connections

$\underline{\underline{\sigma}}$ SA:	$\underline{\text{div}}\,\underline{\underline{\sigma}} + \underline{f}^d = 0 \text{ in } \Omega$ $\underline{\underline{\sigma}}_1 \underline{n} = \underline{F}^d \text{ on } \partial\Omega_F$
$\underline{u}$ CA:	$\underline{u} = \underline{u}^d \text{ on } \partial\Omega_u$ $\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\underline{\text{grad}}}\,\underline{u} + \underline{\underline{\text{grad}}}^T\underline{u})$
Behaviour:	$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}}$

where $\underline{\underline{\underline{\underline{K}}}}$ is the elasticity operator.

This problem can also be written: find $\underline{u}$ and $\underline{\underline{\sigma}}$ such that :

$$\underline{u} \in \mathcal{U}_{ad} \quad ; \quad \underline{\underline{\sigma}} \in \Sigma_{ad} \quad ; \quad \underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}}$$

where $\mathcal{U}_{ad}$ is the kinematic admissibility space:

$$\mathcal{U}_{ad} = \{\underline{u}, \text{ regular } / \quad \underline{u} = \underline{u}^d \text{ on } \partial\Omega_u\}$$

and Σ_{ad} is the static admissibility space:

$$\Sigma_{ad} = \left\{\underline{\underline{\sigma}}, \text{ symmetric } / \quad \underline{\text{div}}\,\underline{\underline{\sigma}} + \underline{f}^d = 0 \text{ in } \Omega \text{ and } \underline{\underline{\sigma}}\underline{n} = \underline{F}^d \text{ on } \partial\Omega_F\right\}$$

For the variational formulations, the following zero kinematicaly and statically spaces can be defined:

$$\mathcal{U}_{ad0} = \{\underline{u}^*, \text{ regular } / \quad \underline{u}^* = 0 \text{ on } \partial\Omega_u\}$$

and

$$\Sigma_{ad0} = \{\underline{\underline{\sigma}}', \text{ symmetric } / \quad \underline{\text{div}}\,\underline{\underline{\sigma}}' = 0 \text{ in } \Omega \text{ and } \underline{\underline{\sigma}}'\underline{n} = 0 \text{ on } \partial\Omega_F\}$$

1.3.2 Variational principle

Writting a global formulation of the equilibrium equation

$$\underline{\text{div}}\,\underline{\underline{\sigma}} + \underline{f}^d = 0, \forall M \in \Omega \Leftrightarrow \int_\Omega (\underline{\text{div}}\,\underline{\underline{\sigma}} + \underline{f}^d)\underline{u}^* d\Omega = 0 \quad , \qquad \forall \underline{u}^* \in \mathcal{U}_{ad0}$$

leads, after integration, to the following :

$\underline{u}$, solution, is kinematicaly admissible and such that:

$$\int_\Omega \text{Tr } (\underline{\underline{\underline{\underline{K}}}}\, \underline{\underline{\varepsilon}}(\underline{u})\underline{\underline{\varepsilon}}(\underline{u}^*))d\Omega - \int_\Omega \underline{f}^d \underline{u}^* dS - \int_{\partial\Omega_F} \underline{F}^d \underline{u}^* dS = 0 \quad , \qquad \forall \underline{u}^* \in \mathcal{U}_{ad0}$$

1.3.3 Energy theorem

The energy partition is made from the error in constituitive relation wich hes to be minimum for a kinematically admissible displacement and a staticaly admissible stress field:

$$\begin{aligned} e(\underline{u}, \underline{\underline{\sigma}}) =& \frac{1}{2}\int_\Omega \big(\underline{\underline{\sigma}} - \underline{\underline{\underline{\underline{K}}}}\, \underline{\underline{\varepsilon}}\big)\underline{\underline{\underline{\underline{K}}}}^{-1}\big(\underline{\underline{\sigma}} - \underline{\underline{\underline{\underline{K}}}}\, \underline{\underline{\varepsilon}}\big)d\Omega \\ =& \frac{1}{2}\int_\Omega \underline{\underline{\underline{\underline{K}}}}^{-1}\underline{\underline{\sigma}}\, \underline{\underline{\sigma}} d\Omega + \frac{1}{2}\int_\Omega \underline{\underline{\underline{\underline{K}}}}^{-1}\underline{\underline{\varepsilon}}\, \underline{\underline{\varepsilon}} d\Omega - \int_\Omega \underline{\underline{\sigma}}\, \underline{\underline{\varepsilon}} d\Omega \quad ; \quad \underline{u} \in \mathcal{U}_{ad}, \underline{\underline{\sigma}} \in \Sigma_{ad} \end{aligned}$$

Developping the last term and using the admissibility conditions gives:

$$\begin{aligned} e(\underline{u}, \underline{\underline{\sigma}}) =& \frac{1}{2}\int_\Omega \underline{\underline{\underline{\underline{K}}}}^{-1}\underline{\underline{\sigma}}\, \underline{\underline{\sigma}} d\Omega - \int_{\partial\Omega_u} \underline{\underline{\sigma}}\underline{n}\; \underline{u}^d dS \\ &+ \frac{1}{2}\int_\Omega \underline{\underline{\underline{\underline{K}}}}^{-1}\underline{\underline{\varepsilon}}\, \underline{\underline{\varepsilon}} d\Omega - \int_\Omega \underline{f}^d \underline{u} d\Omega - \int_{\partial\Omega_F} \underline{F}^d \underline{u} dS \\ =& E_c(\underline{\underline{\sigma}}) + E_p(\underline{u}) \end{aligned}$$

which shows the partition into complementary and potential energy.

With the finite element discrtisation, the theorem of potential enrgy is uised: $\underline{u}$, solution, is kinematicaly admissible and minimize the potential energy that:

$$\underline{v} \in \mathcal{U}_{ad} \longrightarrow E_p(\underline{v}) = \frac{1}{2}\int_\Omega \text{Tr } (\underline{\underline{\underline{\underline{K}}}}\, \underline{\underline{\varepsilon}}(\underline{v})\underline{\underline{\varepsilon}}(\underline{v}))d\Omega - \int_\Omega \underline{f}^d \underline{v} dS - \int_{\partial\Omega_F} \underline{F}^d \underline{v} dS$$

1.3.4 Finite element approximation

$$\underline{u}(M) = \sum_{i=1}^{N} q_i \underline{\varphi}_i(M)$$

$[q]$, such that $\underline{u} \in \mathcal{U}_{ad}$, minimise

$$E_p([q]) = \frac{1}{2}[q]^t[K][q] - [q]^t[F] \qquad \Rightarrow \qquad [K][q] = [F]$$

where the terms of the stiffness matrix $[K]$ are:

$$k_{ij} = \int_\Omega \text{Tr } (\underline{\underline{\underline{\underline{K}}}}\, \underline{\underline{\varepsilon}}(\underline{\varphi}_i)\underline{\underline{\varepsilon}}(\underline{\varphi}_j))d\Omega$$

and the terms of the vector of generalised forces are:

$$f_i = \int_\Omega \underline{f}^d \underline{\varphi}_i dS + \int_{\partial\Omega_F} \underline{F}^d \underline{\varphi}_i dS$$

Chapter 2

Boundary conditions on a solid

2.1 Prescribed forces

2.2 Prescribed conditions on degrees of freedom

2.2.1 General expression of the conditions

The finite element method is based on the minimisation of an energy under condition of admissibility (prescribed displacement, prescribed temperature, ...) that are expressed as conditions on the degrees of freedom. We shall write those conditions under the general following matrix form:

$$[C][q] = [b]$$

where vector $[q]$ is the vector of degrees of freedom, matrix contain the expression of the conditions $[C]$ and vector $[b]$ contains the given prescribed values. Examples are given in the following sections.

The problem to solve is then, find vector $[q]$ such that :

$$[q] \ minimise \ E_p([q]) = \frac{1}{2}[q]^t[K][q] - [q]^t[F]$$

$$and \ satisfies \ [C][q] = [b]$$

2.2.2 Prescribed conditions

In the bar example proposed on figure 2.1, the dsplacement degrees of freedom are submitted to the following conditions:

$$u_1 = 0 \qquad ; \qquad u_3 = u_d$$

The general matrix form condition is then:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[C]} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{[q]} = \underbrace{\begin{bmatrix} 0 \\ u_d \end{bmatrix}}_{[b]}$$

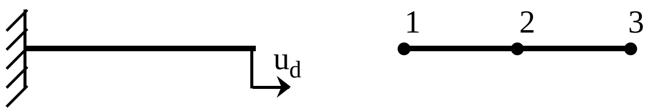

Figure 2.1: Exemple of prescribed conditions on the dof

2.3 Methods

2.3.1 Substitution technique

The substitution technique consists in making operation on the system, in order to substitute selected degrees of freedom by:

- their given value, if their are submitted to prescribed conditions,
- a linear combination of other degree of freedom if their are submitted to linear dependance.

In the case of a $x \times n$ system, for which the first degree of freedom, for example, in prescribed such that $q_1 = b$, it can be sustitute by the given value and the matrix system becomes:

$$\begin{bmatrix} k_{22} & \dots & k_{2j} & \dots & k_{2n} \\ \vdots & & & & \\ k_{i2} & \dots & k_{ij} & \dots & k_{in} \\ \vdots & & & & \\ k_{n2} & \dots & k_{nj} & \dots & k_{nn} \end{bmatrix} \begin{bmatrix} q_2 \\ \vdots \\ q_i \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} f_2 - k_{21} b \\ \vdots \\ f_i - k_{i1} b \\ \vdots \\ f_n - k_{n1} b \end{bmatrix}$$

In the case where the degrees of freedom are linked together by a linear dependance condition:

$$\sum_{k=1}^{n} a_k q_k = b$$

the first degree of freedom q_1, for example, can be substitue, and the matrix system becomes:

$$\begin{bmatrix} k_{22} + k_{21}\frac{a_2}{a_1} & \dots & k_{2j} + k_{21}\frac{a_j}{a_1} & \dots & k_{2n} + k_{21}\frac{a_n}{a_1} \\ \vdots & & & & \\ k_{i2} + k_{i1}\frac{a_2}{a_1} & \dots & k_{ij} + k_{i1}\frac{a_j}{a_1} & \dots & k_{in} + k_{i1}\frac{a_n}{a_1} \\ \vdots & & & & \\ k_{n2} + k_{n1}\frac{a_2}{a_1} & \dots & k_{nj} + k_{n1}\frac{a_j}{a_1} & \dots & k_{nn} + k_{n1}\frac{a_n}{a_1} \end{bmatrix} \begin{bmatrix} q_2 \\ \vdots \\ q_i \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} f_2 - k_{21}\frac{b}{a_1} \\ \vdots \\ f_i - k_{i1}\frac{b}{a_1} \\ \vdots \\ f_n - k_{n1}\frac{b}{a_1} \end{bmatrix}$$

If we consider the previous bar problem proposed on figure 2.1, we have:

$$[K] = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \quad \text{et} \quad [F] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where $k = \frac{ES}{L}$, with L the length of the elements, S the section of the bar and E the Young's modulus.

After sunstitution, the system is reduced to:

$$\begin{bmatrix} 2k \end{bmatrix} \begin{bmatrix} u_2 \end{bmatrix} = \begin{bmatrix} +ku_d \end{bmatrix}$$

which solution is:

$$u_2 = \frac{u_d}{2}$$

that is the exact solution.

Reaction forces The reaction forces can be computed using the equilibrium:

$$[f] = [K][q]$$

in which we only compute the needed forces. In the previous example, this gives:

$$= \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} 0 \\ u_d/2 \\ u_d \end{bmatrix}$$

in which we only compute the reaction forces:

$$F_1 = -\frac{k}{2}u_d \qquad ; \qquad F_3 = \frac{k}{2}u_d$$

2.3.2 Penalisation

method In this approach, we minimise the potential energy to which we have added a enerniti-cal term expressing the condition penalised by the scalar parameter g:

$$E_p(q_i) + \frac{g}{2}\left[[C][q] - [b]\right]^t \left[[C][q] - [b]\right]$$

thus, the more large is g, the more the condition $[C][q] = [b]$ is satisfied.

This approach is very simple because it only consist in adding terms on the diagonal of the matrix:

$$\frac{\partial}{\partial q_i} = 0 \Rightarrow \left[[K] + g[C]^t[C]\right][q] = [F] + g[C]^t[b]$$

This is not an exact approach because the value of g can not be too large due to condiotion-ment problems in the system.

example If we consider the previous bar problem proposed on figure 2.1, we have: When we have:

$$[C]^t[C] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad [C]^t[b] = \begin{bmatrix} 0 \\ 0 \\ u_d \end{bmatrix}$$

The system to be solved is:

$$\begin{bmatrix} k+g & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k+g \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ gu_d \end{bmatrix}$$

whose solution:

$$u_1 = \frac{k}{2(k+g)}u_d \simeq 0 \qquad u_2 = \frac{u_d}{2} \qquad u_3 = \frac{k+2g}{2(k+g)}u_d \simeq u_d$$

is not very far from the exact solution when g is large.

Reaction forces As in the substitution method, the reaction forces can be computed using the equilibrium penalized system:

$$[f] = [K'][q]$$

in which we only compute the needed forces. In the previous example, this gives:

$$= \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \begin{bmatrix} k+g & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k+g \end{bmatrix} \begin{bmatrix} \frac{k}{2(k+g)} u_d \\ \frac{u_d}{2} \\ \frac{k+2g}{2(k+g)} u_d \end{bmatrix}$$

in which we only compute the exact reaction forces:

$$F_1 = -\frac{k}{2} u_d \qquad ; \qquad F_3 = \frac{k}{2} u_d$$

2.3.3 Lagrange multiplier method

In this approach, we look for the extrema of:

$$E_p(q_i) + [\lambda]^t \left[[C][q] - [b] \right]$$

They correspond to :

$$\frac{\partial E_p}{\partial q_i} = 0 \Rightarrow [K][q] - [F] + [\lambda]^t [C] = 0$$

$$\frac{\partial E_p}{\partial \lambda_i} = 0 \Rightarrow [C][q] - [b] = 0$$

so the condition is exactely satisfied.

The linear system to be solved is larger because the Lagrange multipliers are unknown:

$$\begin{bmatrix} [K] & [C]^t \\ [C] & [0] \end{bmatrix} \begin{bmatrix} [q] \\ [\lambda] \end{bmatrix} = \begin{bmatrix} [F] \\ [b] \end{bmatrix}$$

One can show that the lagrange multipliers λ_i are the connection forces needed to prescribed the conditions on the degrees of freedom.

If we consider the problem proposed on figure 2.1, one has to solve:

$$\begin{bmatrix} k & -k & 0 & 1 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & k & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_d \end{bmatrix}$$

which solution is:

$$u_1 = 0 \qquad u_2 = \frac{u_d}{2} \qquad u_3 = u_d \qquad \lambda_1 = \frac{k}{2} u_d \qquad \lambda_2 = -\frac{k}{2} u_d$$

that is the exact solution. One can see that the multipliers λ_i correspond to the opposite of the forces needed to wensure the conditions.

One can notice that the matrix in the solved problem is not positive any more.

2.3.4 Double Lagrange multiplier method

In order to get non null terms on the diagonal in the system obtained by the Lagrange multiplier method, the problem can be rewritten using the following variables:

$$[\lambda] = [\lambda'] + [\lambda''] \qquad ; \qquad [\lambda'] = [\lambda'']$$

And the admissibility condition is written as:

$$[C][q] \pm \alpha([\lambda'] - [\lambda'']) = [b]$$

Thus, the linear system to be solved is larger because the Lagrange multipliers are unknown:

$$\begin{bmatrix} [K] & [C]^t & [C]^t \\ [C] & -\alpha\,[I] & \alpha\,[I] \\ [C] & \alpha\,[I] & -\alpha\,[I] \end{bmatrix} \begin{bmatrix} [q] \\ [\lambda'] \\ [\lambda''] \end{bmatrix} = \begin{bmatrix} [F] \\ [b] \\ [b] \end{bmatrix}$$

2.3.5 Augmented lagragian formulation

2.4 Conclusions

Chapter 3

Connections between two solids

3.1 Local formulation of the connection problem

We are looking for a displacement field $\underline{u}$ and a stress field $\underline{\underline{\sigma}}$. In Ω_1 (resp. Ω_2), $\underline{u}$ is denoted $\underline{u}_1$ (resp. $\underline{u}_2$) and $\underline{\underline{\sigma}}$ is denoted $\underline{\underline{\sigma}}_1$ (resp. $\underline{\underline{\sigma}}_2$). They are subjected to the following conditions in each domain:

$\underline{\underline{\sigma}}_1$ SA:	$\underline{\text{div}}\,\underline{\underline{\sigma}}_1 + \underline{f}_1^d = 0$ in Ω_1 $\underline{\underline{\sigma}}_1 \underline{n}_1 = \underline{F}_1^d$ on $\partial\Omega_{F_1}$	$\underline{\underline{\sigma}}_2$ SA:	$\underline{\text{div}}\,\underline{\underline{\sigma}}_2 + \underline{f}_2^d = 0$ in Ω_2 $\underline{\underline{\sigma}}_2 \underline{n}_2 = \underline{F}_2^d$ on $\partial\Omega_{F_2}$
$\underline{u}_1$ CA:	$\underline{u}_1 = \underline{u}_1^d$ on $\partial\Omega_{u_1}$ $\underline{\underline{\varepsilon}}_1 = \frac{1}{2}(\underline{\underline{\text{grad}}}\,\underline{u}_1 + \underline{\underline{\text{grad}}}^T \underline{u}_1)$	$\underline{u}_2$ CA:	$\underline{u}_2 = \underline{u}_2^d$ on $\partial\Omega_{u_2}$ $\underline{\underline{\varepsilon}}_2 = \frac{1}{2}(\underline{\underline{\text{grad}}}\,\underline{u}_2 + \underline{\underline{\text{grad}}}^T \underline{u}_2)$
Behaviour:	$\underline{\underline{\sigma}}_1 = \underline{\underline{\underline{\underline{K}}}}^1\, \underline{\underline{\varepsilon}}_1$	Behaviour:	$\underline{\underline{\sigma}}_2 = \underline{\underline{\underline{\underline{K}}}}^2\, \underline{\underline{\varepsilon}}_2$

The interface conditions have to be added. They can be static conditions, kinematic condition or behaviour ones. Their study will be the object of the rest of the document. They are expressed by condition on the displacement on the interface and on the forces of each domain on the other. These forces are denoted:

$$\underline{F}_1 = \underline{\underline{\sigma}}_1 \underline{n}_1 \text{ on } \Gamma_1 \qquad ; \qquad \underline{F}_2^\Gamma = \underline{\underline{\sigma}}_2 \underline{n}_2 \text{ on } \Gamma_2$$

The displacement on the interface are the restrictions of the displacement in the domain to the interface:

$$\underline{u}_1^\Gamma = \underline{u}_1|_{\Gamma_1} \qquad ; \qquad \underline{u}_2^\Gamma = \underline{u}_2|_{\Gamma_2}$$

Usually, the interface conditions are expressed in term of an equilibrium equation:

$$\underline{F}_1^\Gamma + \underline{F}_2^\Gamma = 0$$

and a behaviour equation:

$$\underline{F}_2^\Gamma = f(\underline{u}_2^\Gamma - \underline{u}_1^\Gamma) \text{ on } \Gamma$$

where f can be a non linear function.

The displacement jump is equivalent to a deformation for the interface behaviour. It is usually denoted:

$$\underline{\Delta u}^\Gamma = \underline{u}_2^\Gamma - \underline{u}_1^\Gamma$$

3.2 Global formulation of the connection problem

Defining the two subspaces of kinematic admissibility:

$$\mathcal{U}_{ad}^1 = \left\{\underline{u}_1(M), M \in \Omega_1 \text{ such that } \underline{u}_1 = \underline{u}_1^d \text{ on } \partial\Omega_{u_1}\right\}$$
$$\mathcal{U}_{ad}^2 = \left\{\underline{u}_2(M), M \in \Omega_2 \text{ such that } \underline{u}_2 = \underline{u}_2^d \text{ on } \partial\Omega_{u_2}\right\}$$

and their associated zero admisssibility subspaces:

$$\mathcal{U}_{ad0}^1 = \left\{\underline{u}_1^*(M), M \in \Omega_1 \text{ such that } \underline{u}_1^* = 0 \text{ on } \partial\Omega_{u_1}\right\}$$
$$\mathcal{U}_{ad0}^2 = \left\{\underline{u}_2^*(M), M \in \Omega_2 \text{ such that } \underline{u}_2^* = 0 \text{ on } \partial\Omega_{u_2}\right\}$$

the global formulation of the problem is to find $\underline{u}_1 \in \mathcal{U}_{ad}^1$ and $\underline{u}_2 \in \mathcal{U}_{ad}^2$ such that:

$$\begin{aligned}
&\int_{\Omega_1} \text{Tr } (\underline{\underline{\underline{\underline{K}}}}^1 \, \underline{\underline{\varepsilon}}_1 \underline{\underline{\varepsilon}}_1^*) d\Omega - \int_{\Omega_1} \underline{f}_1^d \underline{u}_1^* dS - \int_{\partial\Omega_{F_1}} \underline{F}_1^d \underline{u}_1^* dS + \\
&\int_{\Omega_2} \text{Tr } (\underline{\underline{\underline{\underline{K}}}}^2 \, \underline{\underline{\varepsilon}}_2 \underline{\underline{\varepsilon}}_2^*) d\Omega - \int_{\Omega_2} \underline{f}_2^d \underline{u}_2^* dS - \int_{\partial\Omega_{F_2}} \underline{F}_2^d \underline{u}_2^* dS \\
&\qquad - \int_{\Gamma_1} \underline{F}_1^\Gamma \underline{u}_1^* dS - \int_{\Gamma_2} \underline{F}_2^\Gamma \underline{u}_2^* dS = 0, \forall \underline{u}_1^* \in \mathcal{U}_{ad0}^1, \underline{u}_2^* \in \mathcal{U}_{ad0}^2
\end{aligned}$$

As the domains Ω_1 and Ω_2 are disjointed, $\underline{u}_1$ and $\underline{u}_2$ can be represented by $\underline{u}$ in both domains and thus the kinematic admissibility spaces can be joint:

$$\begin{aligned}
\mathcal{U}_{ad} = \Big\{&\underline{u}(M), M \in \Omega \text{ denoted } \underline{u}_1 \text{ in } \Omega_1 \text{ and } \underline{u}_2 \text{ in } \Omega_2, \\
&\text{such that } \underline{u} = \underline{u}_1^d \text{ on } \partial\Omega_{u_1} \text{ and } \underline{u} = \underline{u}_2^d \text{ on } \partial\Omega_{u_2}\Big\} \\
\mathcal{U}_{ad0} = \Big\{&\underline{u}^*(M), M \in \Omega \text{ denoted } \underline{u}_1^* \text{ in } \Omega_1 \text{ and } \underline{u}_2^* \text{ in } \Omega_2, \\
&\text{such that } \underline{u}^* = 0 \text{ on } \partial\Omega_{u_1} \text{ and } \underline{u}^* = 0 \text{ on } \partial\Omega_{u_2}\Big\}
\end{aligned}$$

and the behaviour operator $\underline{\underline{\underline{\underline{K}}}}$ can represent $\underline{\underline{\underline{\underline{K}}}}^1$ in Ω_1 and $\underline{\underline{\underline{\underline{K}}}}^2$ in Ω_2. Then the formulation can be reduced to find $\underline{u} \in \mathcal{U}_{ad}$ such that:

$$\begin{aligned}
&\int_{\Omega} \text{Tr } (\underline{\underline{\underline{\underline{K}}}} \, \underline{\underline{\varepsilon}} \, \underline{\underline{\varepsilon}}^*) d\Omega - \int_{\Omega} \underline{f}^d \underline{u}^* dS - \int_{\partial\Omega_F} \underline{F}^d \underline{u}^* dS \\
&\qquad - \int_{\Gamma_1} \underline{F}_1^\Gamma \underline{u}_1 dS - \int_{\Gamma_2} \underline{F}_2^\Gamma \underline{u}_2 dS = 0, \forall \underline{u}^* \in \mathcal{U}_{ad0}
\end{aligned}$$

As Γ_1 and Γ_2 represent the same portion of space where exist two surfaces, it is more convinient to use the term Γ to represent the interface and two separate de displacement fields of both sides using the appropriate upperscripts. The formulation becomes to find $\underline{u} \in \mathcal{U}_{ad}$ such that:

$$\int_{\Omega} \text{Tr } (\underline{\underline{\underline{\underline{K}}}} \, \underline{\underline{\varepsilon}} \, \underline{\underline{\varepsilon}}^*) d\Omega - \int_{\Omega} \underline{f}^d \underline{u}^* dS - \int_{\partial\Omega_F} \underline{F}^d \underline{u}^* dS - \int_{\Gamma} (\underline{F}_1^\Gamma \underline{u}_1^* + \underline{F}_2^\Gamma \underline{u}_2^*) dS = 0, \forall \underline{u}^* \in \mathcal{U}_{ad0}$$

As the equilibrium of the interface is always satisfied, the formulation becomes to find $\underline{u} \in \mathcal{U}_{ad}$ such that:

$$\int_\Omega \text{Tr }(\underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{\varepsilon}}^*)d\Omega - \int_\Omega \underline{f}^d \underline{u}^* dS - \int_{\partial\Omega_F} \underline{F}^d \underline{u}^* dS - \int_\Gamma \underline{F}_2^\Gamma(\underline{u}_2^* - \underline{u}_1^*)dS = 0, \forall \underline{u}^* \in \mathcal{U}_{ad0}$$

The expression of the last term will depend on the behaviour of the interface. In the following, the upperscript Γ is omitted when not necessary:

$$\int_\Omega \text{Tr }(\underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{\varepsilon}}^*)d\Omega - \int_\Omega \underline{f}^d \underline{u}^* dS - \int_{\partial\Omega_F} \underline{F}^d \underline{u}^* dS - \int_\Gamma \underline{F}_2(\underline{u}_2^* - \underline{u}_1^*)dS = 0, \forall \underline{u}^* \in \mathcal{U}_{ad0}$$

If one want to built the associated energy theorems, the form of the behaviour equation on the has to be defined first.

3.3 Local directions

Many connection behaviours depends on the local direction on the connecting zone Γ. The local orientation is given by the normal $\underline{n}$ which will here be chosen as the local outward normal of solid Ω_1 which is opposite to the outward normal of solide Ω_2:

$$\underline{n} = \underline{n}_1 = -\underline{n}_2$$

The normal displacements are scalar variable wich represent the displacements in the directioln of the normal:

$$u_{n_1} = \underline{u}_1.\underline{n} \qquad \text{and} \qquad u_{n_2} = \underline{u}_2.\underline{n}$$

The normal $\underline{n}$ have been chosen equal to $\underline{n}_1$ in order that a positive value of $(u_{n_2} - u_{n_2})$ correspondans to a separation of the two structures.

The normal force have the same definition:

$$F_{n_1} = \underline{F}_1.\underline{n} \qquad \text{and} \qquad F_{n_2} = \underline{F}_2.\underline{n}$$

The tangential displacement is the vector of displacement in the tangential plane and is the complementary part of the normal displacement:

$$\underline{u}_{t_1} = \underline{u}_1 - u_{n_1}\underline{n} \qquad \text{and} \qquad \underline{u}_{t_2} = \underline{u}_2 - u_{n_2}\underline{n}$$

and the tangential forces have the same definition:

$$\underline{F}_{t_1} = \underline{F}_1 - F_{n_1}\underline{n} \qquad \text{and} \qquad \underline{F}_{t_2} = \underline{F}_2 - F_{n_2}\underline{n}$$

3.4 Exemples

3.4.1 Perfect connection

In the case of perfect connection, the behaviour equation is reduced to the kinematical constrain:

$$\underline{u}_2 - \underline{u}_1 = 0 \text{ on } \Gamma$$

and is thus added in the subspaces of admissibility:

$$\mathcal{U}'_{ad} = \Big\{ \underline{u}(M), M \in \Omega \text{ denoted } \underline{u}_1 \text{ in } \Omega_1 \text{ and } \underline{u}_2 \text{ in } \Omega_2, \\ \text{such that } \underline{u} = \underline{u}_1^d \text{ on } \partial\Omega_{u_1} \text{ and } \underline{u} = \underline{u}_2^d \text{ on } \partial\Omega_{u_2}, \\ \text{and } \underline{\Delta u} = \underline{u}_2 - \underline{u}_1 = 0 \text{ on } \Gamma \Big\}$$

$$\mathcal{U}'_{ad0} = \Big\{ \underline{u}^*(M), M \in \Omega \text{ denoted } \underline{u}_1^* \text{ in } \Omega_1 \text{ and } \underline{u}_2^* \text{ in } \Omega_2, \\ \text{such that } \underline{u}^* = 0 \text{ on } \partial\Omega_{u_1} \text{ and } \underline{u}^* = 0 \text{ on } \partial\Omega_{u_2}, \\ \text{and } \underline{\Delta u}^* = \underline{u}_2^* - \underline{u}_1^* = 0 \text{ on } \Gamma \Big\}$$

The formulation becomes to classicaly find $\underline{u} \in \mathcal{U}'_{ad}$, such that

$$\int_\Omega \text{Tr}\,(\underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{\varepsilon}}^*)d\Omega - \int_\Omega \underline{f}^d \underline{u}^* dS - \int_{\partial\Omega_F} \underline{F}^d \underline{u}^* dS = 0, \forall \underline{u}^* \in \mathcal{U}'_{ad0}$$

which the formultion obtained on one single part and that's exactely what it is as the two parts are perfectly connected.

The best way to assume this kinematic condition is either to use matching meshes on both sides. It is then easy to prescribed conditions coupling degrees of freedom couple of nodes by couple of nodes using, for example, the lagrange multiplier method as shown is section **??**. It is also possible to merge matching nodes in order to get one single mesh. If the meshes are not the same, special technique are to be used. They are studied in section 6.

exemple: If we consider the problem proposed on figure @@@, the relations corresponding to the prescribed displacement on u_1 and the connection $u_2 = u_3$ are:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using the Lagrange multiplier method leads to the following system:

$$\begin{bmatrix} k & -k & 0 & 0 & 1 & 0 \\ -k & k & 0 & 0 & 0 & 1 \\ 0 & 0 & k & -k & 0 & -1 \\ 0 & 0 & -k & k & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_3 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F \\ 0 \\ 0 \end{bmatrix}$$

which solution is:

$$u_1 = 0 \qquad u_2 = u_3 = \frac{F}{k} \qquad u_4 = \frac{2F}{k} \qquad \lambda_1 = F \qquad \lambda_2 = -F$$

that is the exact solution where the multiplier λ_2 correspond to the opposite of the action of the left part on the right one.

One can also use such a condition with a non relative displacement in order to prescibed a prestrain for example:

$$\underline{\Delta u} = \underline{\Delta}_d \text{ on } \Gamma$$

3.4.2 Bilateral contact

In case of bilateral contact conditions, the behaviour depends on the direction of the normal $\underline{n}$ and is separated in a statical condition in the tangential direction and a kinamatical one in the normal direction:

$$\underline{F}_{t_2} = \underline{F}_{t_1} = 0 \qquad \text{and} \qquad u_{n_2} - u_{n_1} = 0$$

The static condition is added in the formulation that becomes becomes to find $\underline{u} \in \mathcal{U}_{ad}$ such that:

$$\int_\Omega \text{Tr}\,(\underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{\varepsilon}}^*)d\Omega - \int_\Omega \underline{f}^d\underline{u}^* dS - \int_{\partial\Omega_F} \underline{F}^d\underline{u}^* dS - \int_\Gamma F_{n_2}(u^*_{n_2} - u^*_{n_1})dS = 0, \forall \underline{u}^* \in \mathcal{U}_{ad0}$$

The normal kinematic condition is added in the subspaces of admissibility:

$$\begin{aligned}
\mathcal{U}''_{ad} = \Big\{\underline{u}(M), M \in \Omega \text{ denoted } \underline{u}_1 \text{ in } \Omega_1 \text{ and } \underline{u}_2 \text{ in } \Omega_2,\\
\text{such that } \underline{u} = \underline{u}_1^d \text{ on } \partial\Omega_{u_1} \text{ and } \underline{u} = \underline{u}_2^d \text{ on } \partial\Omega_{u_2},\\
\text{and } \underline{\Delta u}.\underline{n} = u_{n_2} - u_{n_1} = 0 \text{ on } \Gamma\Big\}\\
\mathcal{U}''_{ad0} = \Big\{\underline{u}^*(M), M \in \Omega \text{ denoted } \underline{u}_1^* \text{ in } \Omega_1 \text{ and } \underline{u}_2^* \text{ in } \Omega_2,\\
\text{such that } \underline{u}^* = 0 \text{ on } \partial\Omega_{u_1} \text{ and } \underline{u}^* = 0 \text{ on } \partial\Omega_{u_2},\\
\text{and } \underline{\Delta u}^*.\underline{n} = u^*_{n_2} - u^*_{n_1} = 0 \text{ on } \Gamma\Big\}
\end{aligned}$$

The formulation becomes to classicaly find $\underline{u} \in \mathcal{U}''_{ad}$, such that

$$\int_\Omega \text{Tr}\,(\underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{\varepsilon}}^*)d\Omega - \int_\Omega \underline{f}^d\underline{u}^* dS - \int_{\partial\Omega_F} \underline{F}^d\underline{u}^* dS = 0, \forall \underline{u}^* \in \mathcal{U}''_{ad0}$$

The best way to assume this normal kinematic condition is to use matching meshes on both sides. It is then easy to prescribed conditions coupling degrees of freedom corresponding to the normal displacement couple of nodes by couple of nodes using, for example, the lagrange multiplier method as shown is section **??**. If the meshes are not the same, special technique are to be used. They are studied in section 6.

Unilateral contact and friction conditions are studied in section 5.

3.5 Conclusions

Chapter 4

Elastic connections

4.1 Origins

- composites
- adhesive layers
- elastic joints

4.2 Formulation

The behaviour equation of the elastic interface is:

$$\underline{F}_1 = -\underline{F}_2 = \underline{\underline{k}}\ (\underline{u}_2 - \underline{u}_1)$$

For isotropic elastic layers, the interface behaviour will usually take the following form:

$$\underline{\underline{k}} = \begin{bmatrix} k_n & 0 & 0 \\ 0 & k_t & 0 \\ 0 & 0 & k_t \end{bmatrix}_{(\underline{n},\underline{t}_1,\underline{t}_2)}$$

where the stiffness can be evaluated from the elastic parameters of the interface layer:

$$k_n \simeq \frac{E}{e} \qquad \text{and} \qquad k_t \simeq \frac{G}{e}$$

where e is the thickness of the layer.

Using the behaviour equation, the formulation becomes to find $\underline{u} \in \mathcal{U}_{ad}$ such that:

$$\int_\Omega \text{Tr}\ (\underline{\underline{\underline{\underline{K}}}}\ \underline{\underline{\varepsilon}}\ \underline{\underline{\varepsilon}}^*) d\Omega - \int_\Omega \underline{f}^d \underline{u}^* dS - \int_{\partial\Omega_F} \underline{F}^d \underline{u}^* dS - \int_\Gamma -\underline{\underline{k}}\ (\underline{u}_2 - \underline{u}_1)(\underline{u}_2^* - \underline{u}_1^*) dS = 0, \forall \underline{u}^* \in \mathcal{U}_{ad0}$$

that can be written as:

$$\int_\Omega \text{Tr}\ (\underline{\underline{\underline{\underline{K}}}}\ \underline{\underline{\varepsilon}}\ \underline{\underline{\varepsilon}}^*) d\Omega + \int_\Gamma \underline{\underline{k}}\ \underline{\Delta u}\ \underline{\Delta u}^* dS - \int_\Omega \underline{f}^d \underline{u}^* dS - \int_{\partial\Omega_F} \underline{F}^d \underline{u}^* dS = 0, \forall \underline{u}^* \in \mathcal{U}_{ad0}$$

The second term will give an additional stiffness term in the finite element formulation of the problem.

4.3 Energy theorems

As the interface behaviour is expressed in term of a behaviour equation, it as to be taken into account in the error in constituive relation:

$$
\begin{aligned}
e(\underline{u},\underline{\underline{\sigma}}) &= \frac{1}{2}\int_\Omega (\underline{\underline{\sigma}} - \underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}})\underline{\underline{\underline{\underline{K}}}}^{-1}(\underline{\underline{\sigma}} - \underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}})d\Omega + \frac{1}{2}\int_\Gamma (\underline{F}_2 + \underline{\underline{k}}\,\Delta\underline{u})\underline{\underline{k}}^{-1}(\underline{F}_2 + \underline{\underline{k}}\,\Delta\underline{u})dS \\
&= \frac{1}{2}\int_\Omega \underline{\underline{\underline{\underline{K}}}}^{-1}\underline{\underline{\sigma}}\,\underline{\underline{\sigma}}d\Omega + \frac{1}{2}\int_\Omega \underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{\varepsilon}}d\Omega - \int_\Omega \underline{\underline{\sigma}}\,\underline{\underline{\varepsilon}}d\Omega \\
&\quad + \frac{1}{2}\int_\Gamma \underline{\underline{k}}^{-1}\underline{F}_2\underline{F}_2 dS + \frac{1}{2}\int_\Gamma \underline{\underline{k}}\,\Delta\underline{u}\,\Delta\underline{u}dS + \int_\Gamma \underline{F}_2\Delta\underline{u}dS \\
&= \frac{1}{2}\int_\Omega \underline{\underline{\underline{\underline{K}}}}^{-1}\underline{\underline{\sigma}}\,\underline{\underline{\sigma}}d\Omega + \frac{1}{2}\int_\Gamma \underline{\underline{k}}^{-1}\underline{F}_2\underline{F}_2 dS + \frac{1}{2}\int_\Omega \underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{\varepsilon}}d\Omega + \frac{1}{2}\int_\Gamma \underline{\underline{k}}\,\Delta\underline{u}\,\Delta\underline{u}dS \\
&\quad - \int_{\partial\Omega_u} \underline{\underline{\sigma}}\underline{n}\,\underline{u}^d dS - \int_\Omega \underline{f}^d\underline{u}d\Omega - \int_{\partial\Omega_F} \underline{F}^d\underline{u}dS - \int_\Gamma \underline{F}_2\Delta\underline{u}dS + \int_\Gamma \underline{F}_2\Delta\underline{u}dS
\end{aligned}
$$

As the last two terms disapear, the potential energy is :

$$
E_p(\underline{u}) = \frac{1}{2}\int_\Omega \underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{\varepsilon}}d\Omega + \frac{1}{2}\int_\Gamma \underline{\underline{k}}\,\Delta\underline{u}\,\Delta\underline{u}dS - \int_\Omega \underline{f}^d\underline{u}d\Omega - \int_{\partial\Omega_F} \underline{F}^d\underline{u}dS
$$

wich corresponds to the previous variationnal formulation.

Another way to obtain this potential energy is to consider that the interface is an interior part of the structure and thus that Γ is not a part of the boundary of Ω. The deformation energy of the interface can then simply be added to the potential energy of the structure.

4.4 Interface elements

In the finite element framework, the surface energy due to the elastic connection is introduced by using *interface element*. They are lineic element (lines in 2D) or surface element (triangles or quadrangles in 3D) presenting double nodes (figure 4.1).

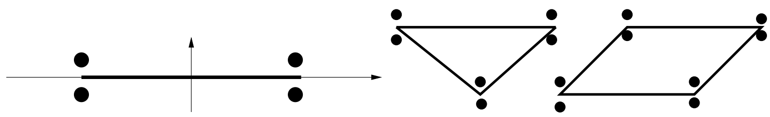

Figure 4.1: 2D and 3D interface elements

On such elemnts, the discretised kinematical variable is the jump in displacement:

$$
\Delta\underline{u}(M) = \sum_{i=1}^{N} \Delta q_i \varphi_i(M)
$$

for example, on the 2D linear interface element (figure 4.2) there are four degrees of freedom:

$$
[\Delta q] = \begin{bmatrix} \Delta q_x^i \\ \Delta q_y^i \\ \Delta q_x^j \\ \Delta q_x^j \end{bmatrix}
$$

and the stiffness operator if a 4×4 matrix:

$$
E_d([\Delta q]) = \frac{1}{2}[\Delta q]^t[K][\Delta q]
$$

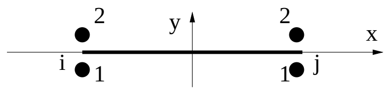

Figure 4.2: 2D linear interface elements

Using the changes in variables:

$$\Delta q_x^i = q_x^{i2} - q_x^{i1} \qquad ; \qquad \Delta q_y^i = q_y^{i2} - q_y^{i1}$$

the vector of degrees of freedom becomes:

$$[\Delta q]^t = \begin{bmatrix} q_x^{i2} & q_x^{i1} & q_y^{i2} & q_y^{i1} & q_x^{j2} & q_x^{j1} & q_y^{j2} & q_y^{j1} \end{bmatrix}$$

and the stiffness operator becomes a 8×8 matrix that can be assembled in the global stiffness matrix as the used degrees of freedom match the degrees of freedom of the rest of the volumic elements.

4.5 Extensions

Chapter 5

Unilateral contact and friction

5.1 Contact non linearities

In the structural mechanic problems, their are three types of non linearities :

- the *behaviour* non linearities which are due to the non linear aspect of the constituive relation : plasticity, viscoplasticity, damage, ...
- the *geometrical* non linearities which are to be taken into account in the case of large deformation of large displacement i.e. when the current configuration can not be considered as the same as the initial configuration.
- the *contact* non linearities which are associated to the fact that, in the connection between two solids, opening and frictional sliding can occur. Their are separated into two groups : *unilateral contact* non linearities and *friction* ones.

The non linearities due to contact are the strongest one because their are associated to strong discontinuities in the behaviour :

- transition between opening and contact
- tansition between sticking and sliding

5.2 Unilateral contact without friction

5.2.1 Local behaviour

We only consider here frictional contact condition with no adhesion. The local behaviour have to express the following phenomenological condiderations:

- there is non penetration of one solid into the other one,
- there is no adhesion,
- there is no friction,
- on one same point, there can not be opening and closure at the same time

As the equilibrium is always satisfied, the local behaviour will be expressed on the normal and tangential contact forces that are defined by:

$$F_n = F_{n_1} = -F_{n_2} \quad \text{and} \quad \underline{F}_t = \underline{F}_{t_1} = -\underline{F}_{t_2}$$

The relative displacement is expressed by is normal and tangential parts:

$$\Delta u_n = \underline{\Delta u}.\underline{n} = u_{n_2} - u_{n_1} \quad \text{and} \quad \underline{\Delta u}_t = \underline{\Delta u} - \Delta u_n \underline{n} = \underline{u}_{t_2} - \underline{u}_{t_1}$$

The previous consideration are then expressed by the following condition:

$\underline{F}_t = 0$	no friction
$F_n \leq 0$	no adhesion
$\Delta u_n \geq 0$	no penetration
$F_n \Delta u_n = 0$	no simultaneous closure and opening

The first two equations are statical admissibility conditions. The third one is a kinematical admissiblity condition. As it is negative, F_n is called *contact pressure*. As it is always positive Δu_n is called *contact opening*.

The last one expressed the fact that one can not prescribe a condition on the normal forces and on the normal displacement jump at the same time. It is called *complementary condition* and is a constituitive relation has it is a relation between displacement and forces. It says that:

- in case of clossure : the contact pressure can be non zero but the opening is nul,
- in case of opening : the contact opening can be non zero but the contact pressure is nul.

The local contact conditions are thus of three types : statical, kinematical and consituive relation conditions. They are then to be taken into account at different stages of the construction of the global formulation.

5.2.2 Global formulation

The kinematical admissibility conditions are taken into account by searching the solution in the following convex subspace of the admissible space:

$$\mathcal{U}_{ad}^c = \{\underline{u} \in \mathcal{U}_{ad} / \Delta u_n = \underline{\Delta u}.\underline{n} = u_{n_2} - u_{n_1} \geq 0\}$$

using search functions in the associated zero admissibility convex:

$$\mathcal{U}_{ad0}^c = \left\{\underline{u}^* \in \mathcal{U}_{ad0} / \Delta u_n^* = \underline{\Delta u}^*.\underline{n} = u_{n_2}^* - u_{n_1}^* \geq 0\right\}$$

The statically admissibility conditions are taken into account by searching the solution in the following convex subspace of the admissible space:

$$\Sigma_{ad}^c = \left\{\underline{\underline{\sigma}} \in \Sigma_{ad} / F_n = F_{n_1} = -F_{n_2} \leq 0 \text{ and } \underline{F}_t = \underline{F}_{t_1} = -\underline{F}_{t_2} = 0\right\}$$

Variational formulation in displacement Introducing the non friction static condition the variational formulation becomes becomes to find $\underline{u} \in \mathcal{U}_{ad}^c$ such that:

$$\int_\Omega \text{Tr}\,(\underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{\varepsilon}}^*)d\Omega - \int_\Omega \underline{f}^d \underline{u}^* dS - \int_{\partial\Omega_F} \underline{F}^d \underline{u}^* dS - \int_\Gamma F_{n_2}(u_{n_2}^* - u_{n_1}^*)dS = 0, \forall \underline{u}^* \in \mathcal{U}_{ad0}^c$$

The non-penetration condition and the no-adhesion condition can fust say that the last integral is positive. The formulation then becomes a variational inequation that consist in finding $\underline{u} \in \mathcal{U}_{ad}^c$ such that:

$$\int_\Omega \text{Tr}\,(\underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{\varepsilon}}^*)d\Omega - \int_\Omega \underline{f}^d \underline{u}^* dS - \int_{\partial\Omega_F} \underline{F}^d \underline{u}^* dS \geq 0, \forall \underline{u}^* \in \mathcal{U}_{ad0}^c$$

Due to its inequation form, this formulation is not usable for finding an approximate solution (using the finite element method for example).

Energy theorem As there are two behaviour equation the error in constituitive relation has two terms and one must remark that the second one is always positive when the $\underline{u}$ and $\underline{\underline{\sigma}}$ are admissible:

$$e(\underline{u},\underline{\underline{\sigma}}) = \frac{1}{2}\int_\Omega (\underline{\underline{\sigma}} - \underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}})\underline{\underline{\underline{\underline{K}}}}^{-1}(\underline{\underline{\sigma}} - \underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}})d\Omega + \int_\Gamma F_{n_2}\Delta u_n dS \quad \underline{u} \in \mathcal{U}^c_{ad}, \underline{\underline{\sigma}} \in \Sigma^c_{ad}$$

Developping the first term gives:

$$e(\underline{u},\underline{\underline{\sigma}}) = \frac{1}{2}\int_\Omega \underline{\underline{\underline{\underline{K}}}}^{-1}\underline{\underline{\sigma}}\,\underline{\underline{\sigma}}d\Omega + \frac{1}{2}\int_\Omega \underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{\varepsilon}}d\Omega - \int_{\partial\Omega_u} \underline{\underline{\sigma n}}\,\underline{u}^d dS - \int_\Omega \underline{f}^d\underline{u}d\Omega - \int_{\partial\Omega_F} \underline{F}^d\underline{u}dS$$
$$- \int_\Gamma \underline{F}_2\underline{\Delta u}dS + \int_\Gamma F_{n_2}\Delta u_n$$

and introducing the no friction condition leads to:

$$e(\underline{u},\underline{\underline{\sigma}}) = \frac{1}{2}\int_\Omega \underline{\underline{\underline{\underline{K}}}}^{-1}\underline{\underline{\sigma}}\,\underline{\underline{\sigma}}d\Omega + \frac{1}{2}\int_\Omega \underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{\varepsilon}}d\Omega - \int_{\partial\Omega_u} \underline{\underline{\sigma n}}\,\underline{u}^d dS - \int_\Omega \underline{f}^d\underline{u}d\Omega - \int_{\partial\Omega_F} \underline{F}^d\underline{u}dS$$
$$- \int_\Gamma F_{n_2}\Delta u_n + \int_\Gamma F_{n_2}\Delta u_n$$

so that the last two terms disappear. Thus the potential energy theorem is: the solution of the problem $\underline{u} \in \mathcal{U}^c_{ad}$ minimise:

$$\underline{v} \in \mathcal{U}^c_{ad} \longrightarrow Ep(\underline{v}) = +\frac{1}{2}\int_\Omega \underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}}(\underline{v})\,\underline{\underline{\varepsilon}}(\underline{v})d\Omega - \int_\Omega \underline{f}^d\underline{v}d\Omega - \int_{\partial\Omega_F} \underline{F}^d\underline{v}dS$$

where the potential energy has the classical form. This is an *optimisation problem* i.e. the minimisation of a function under constraints of inequality. There are many different methods for solving such optimisation problems. Some of them are presented in the following sections. Such an optimisation problem is known as *Quadratic Programming Optimisation Problem* as the objective function is quadratic.

5.2.3 Discretisation

The best way to ensure the kinematical admissibility is to prescribed the inequality conditions on couple of nodes using comptaible meshes. For example, on figure 5.1 the non penetration condition $\Delta u_n \geq 0$ can be written :

$$v_{i2} - v_{i1} \geq 0 \qquad \text{and} \qquad v_{j2} - v_{j1} \geq 0$$

that can be written in a matrix form of this type:

$$[C][q] \geq [b]$$

and one can see that the discrete condition is equivalent to the continuous one.

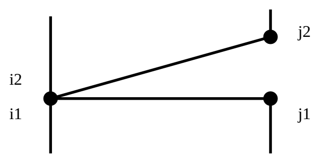

Figure 5.1: Linear discretisation

It is important to notice that this equivalence is not ensured when the discretisation is not linear as one can see one figure 5.2.

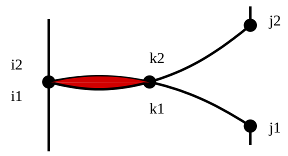

Figure 5.2: Non Linear discretisation

5.2.4 Resolution methods

Status method In the field of contact mechanism, this method is known as *Status Method* or *Active Constrains Method*. It is based on the *Active Set Method* known in the field of optimisation.

Projection methods

Other methods

5.2.5 Regularisation of the contact law

5.3 Friction

Chapter 6

Incompatible connections

6.1 Ponctual connection

6.2 Mean connection

6.3 Connections between different models

Chapter 7

Domain decomposition for assemblies

Appendix A

Exercices

A.1 Boundary conditions on a lattice

A.1.1 Text

We consider the three bar lattice presented on figure A.1 (left).

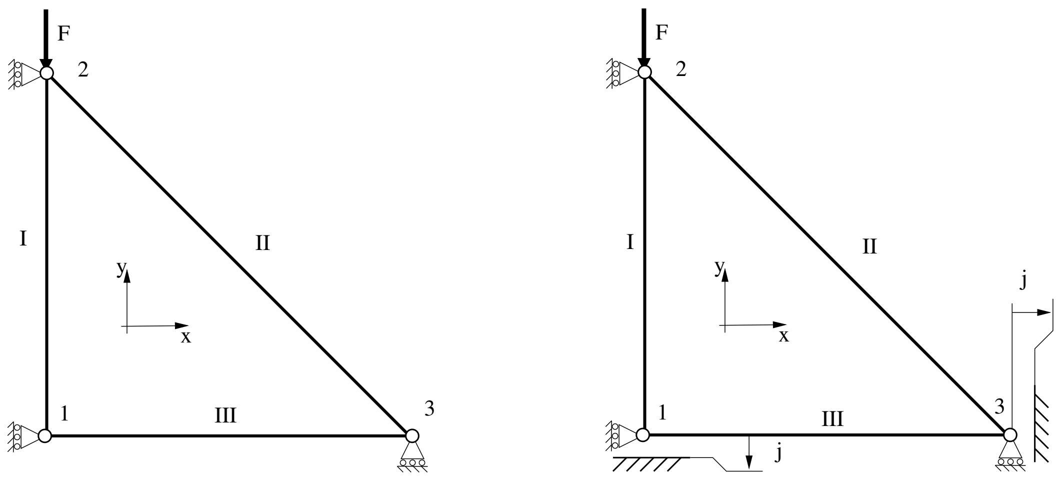

Figure A.1: Lattice : isostatical (left) and with unilateral supports (right)

u_{x_i} and u_{y_i} denote the values of displacement of node i. The geometrical and material caracteristics of the bars are such that the stiffness matrix of the bars are:

$$[K_I] = \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k & 0 & -k \\ 0 & 0 & 0 & 0 \\ 0 & -k & 0 & k \end{bmatrix} \end{array} \begin{array}{c} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \end{array} , [K_{II}] = \begin{bmatrix} k & -k & -k & k \\ -k & k & k & -k \\ -k & k & k & -k \\ k & -k & -k & k \end{bmatrix} , \begin{array}{c} u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \end{array} ,$$

$$[K_{III}] = \begin{bmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} u_{x_1} \\ u_{y_1} \\ u_{x_3} \\ u_{y_3} \end{array}$$

1. Assemble the stiffness matrix of the lattice as well as the vector of forces.

2. Write de boundary conditions presented on the left figure in a matrix form.
3. Build the given system when those condition are taken into account by *substitution*.
4. Solve the problem for the case presented on the left figure.
5. We now also lock the degree of freedom u_{x_3}. Solve by taking into account this new condition with the lagrange multiplier method applied two the system built at question 3. Give the reaction force in direction x on the support of node 3.
6. We now add unilateral frictionless supports with gaps on node 1 and 3, such as presented on the right figure. The gap is the same on the two support and denoted j. It is such that $j = \frac{3F}{2k}$.
   (a) Write de unilateral conditions on the degrees of freedom and on the reaction forces.
   (b) Solve the system by using the *status method*. In this approach, the conditions on the dof will be prescribed using the lagrange multiplier method. Gibe the displecement of the nodes and the reaction force in the supports for the solution.

Note: we shall recall that the lagrange multiplier is the opposite of the reaction force.

A.1.2 Correction

1. After assembly, the system to be solved is:

$$\begin{bmatrix} k & 0 & 0 & 0 & -k & 0 \\ 0 & k & 0 & -k & 0 & 0 \\ 0 & 0 & k & -k & -k & k \\ 0 & -k & -k & 2k & k & -k \\ -k & 0 & -k & k & 2k & -k \\ 0 & 0 & k & -k & -k & k \end{bmatrix} \begin{Bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -F \\ 0 \\ 0 \end{Bmatrix}$$

2. The boundary conditions can be written in a matrix form:

$$\left\{ \begin{array}{l} u_{x_1} = 0 \\ u_{x_2} = 0 \\ u_{y_3} = 0 \end{array} \right. \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

3. When those conditions are taken into account using the substitution technique, there are only three unknowns left and to system to be solved is:

$$\begin{bmatrix} k & -k & 0 \\ -k & 2k & k \\ 0 & k & 2k \end{bmatrix} \begin{Bmatrix} u_{y_1} \\ u_{y_2} \\ u_{x_3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -F \\ 0 \end{Bmatrix}$$

4. The resolution of the systel gives:

$$u_{y_1} = \frac{F}{k} \quad ; \quad u_{y_2} = -\frac{2F}{k} \quad ; \quad u_{x_3} = -\frac{2F}{k}$$

5. When adding the condition $u_{x_3} = 0$, using the lagrange multiplier method, the system becomes:
$$\begin{bmatrix} k & -k & 0 & 0 \\ -k & 2k & k & 0 \\ 0 & k & 2k & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u_{y_1} \\ u_{y_2} \\ u_{x_3} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} 0 \\ -F \\ 0 \\ 0 \end{Bmatrix}$$
which solution is:
$$u_{y_1} = \frac{F}{k} \quad ; \quad u_{y_2} = -\frac{F}{k} \quad ; \quad u_{x_3} = 0 \quad ; \quad \lambda = F$$
Reaction on node 3 in direction x is then: $F_{x_3} = -\lambda = -F$.

6. We add unilateral frictionless supports on nodes 1 and 3 with gap j such that$j = \frac{3F}{2k}$.
   (a) The unilateral conditions are:
$$\left\{ \begin{array}{l} u_{y_1} \geq -j \\ u_{x_3} \leq j \end{array} \right. \qquad \text{et} \qquad \left\{ \begin{array}{l} F_{y_1} \geq 0 \\ F_{x_3} \leq 0 \end{array} \right.$$
   (b) Solution by the status method:

Stage 1 Solve using the strict conditions:
$$\left\{ \begin{array}{l} u_{y_1} = -j \\ u_{x_3} = j \end{array} \right. \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{y_1} \\ u_{y_2} \\ u_{x_3} \end{Bmatrix} = \begin{Bmatrix} -j \\ j \end{Bmatrix}$$
using the Lagrange multiplier technique. The system is:
$$\begin{bmatrix} k & -k & 0 & 1 & 0 \\ -k & 2k & k & 0 & 0 \\ 0 & k & 2k & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{y_1} \\ u_{y_2} \\ u_{x_3} \\ \lambda_1 \\ \lambda_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -F \\ 0 \\ -j \\ j \end{Bmatrix}$$
which solution is:
$$u_{y_1} = -j = -\frac{3F}{2k} \quad ; \quad u_{y_2} = -\frac{2F}{k} \quad ; \quad u_{x_3} = j = \frac{3F}{2k} \quad ; \quad \lambda_1 = -\frac{F}{2} \quad ; \quad \lambda_3 = -F$$
We check the conditions on the forces:
$$\left\{ \begin{array}{ll} F_{y_1} = -\lambda_1 \geq 0 & \text{(satisfied condition, will stay)} \\ F_{x_3} = -\lambda_3 \geq 0 & \text{(non satisfied condition, should be suppressed)} \end{array} \right.$$

Stage 2 Solve the system prescribing the condition that where keeped at the end of last stage:
$$\left\{ \; u_{y_1} = -j \right. \Rightarrow \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{y_1} \\ u_{y_2} \\ u_{x_3} \end{Bmatrix} = \begin{Bmatrix} -j \end{Bmatrix}$$
using the Lagrange multiplier technique. The system is:
$$\begin{bmatrix} k & -k & 0 & 1 \\ -k & 2k & k & 0 \\ 0 & k & 2k & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{y_1} \\ u_{y_2} \\ u_{x_3} \\ \lambda_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -F \\ 0 \\ -j \end{Bmatrix}$$

which solution is:

$$u_{y_1} = -j = -\frac{3F}{2k} \quad ; \quad u_{y_2} = -\frac{5F}{3k} \quad ; \quad u_{x_3} = \frac{5F}{6k} \quad ; \quad \lambda_1 = -\frac{F}{6}$$

We check the conditions on the forces:

$$\begin{cases} F_{y_1} = -\lambda_1 = \frac{F}{6} \geq 0 & \text{OK} \\ u_{y_1} = -j & \text{OK} \\ u_{x_3} = -\frac{F}{6} \leq j & \text{OK} \end{cases}$$

The calculated solution is then the right one.

A.2 Incompatible meshes

A.2.1 Text

We want to make a displacement connection between two meshes along an interface. The interface as a length $2e$ and is oriented by direction x. The position of the nodes on the interface is parametrized by position x and the origin is in the center of the interface. The mesh situated at le left of the interface is denoted I and the one at the right II.

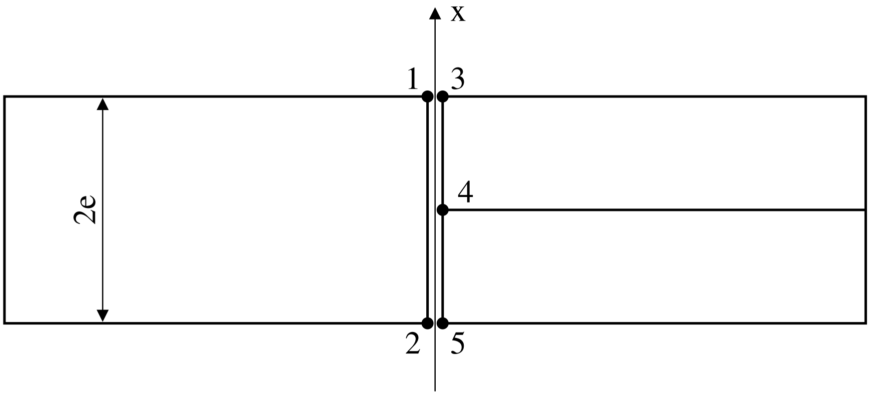

Figure A.2: Incompatibles meshes

For simplification, we are only interested in the connection of one term of displacement denoted u. $u^I(x)$ (resp $u^{II}(x)$) is the expression of this term on mesh I (resp II). On the interface, there is just only one linear element on mesh I wich degree of freedom are denoted u_1 are u_2. Mesh II is made of two linear elements which degrees of freedom are u_3, u_4 et u_5 (see figure A.2). The analytical expression of the displacements $u^I(x)$ et de $u^{II}(x)$ are then:

$$u^I(x) = \frac{u_1 - u_2}{2e}\,x + \frac{u_1 + u_2}{2}, \quad \forall x \in [-e, e] \qquad ; \qquad u^{II}(x) = \begin{cases} \dfrac{u_3 - u_4}{e}x + u_4 \quad , & \text{if } x \in [0, e] \\ \dfrac{u_4 - u_5}{e}x + u_4 \quad , & \text{if } x \in [-e, 0] \end{cases}$$

1. Explain the notions of master mesh and slave mesh.
2. Explain the notion of *connection at a mean sens*.
3. Gives the conditions that have to be prescribe on degree of freedom u_1, u_2, u_3, u_4 et u_5 depending on the type of connection
   (a) ponctual connection for which, mesh I is the master mesh.
   (b) ponctual connection for which, mesh II is the master mesh.
   (c) connection at a mean sens allowing the transmition of constant forces.
   (d) connection at a mean sens allowing the transmition of linear forces.

A.2.2 Correction

1. See lecture notes
2. See lecture notes
3. Conditions on the degrees of freedom
   (a) ponctual connection for which, mesh I is the master mesh.

$$\begin{cases} u_1 - u_3 & = 0 \\ u_2 - u_5 & = 0 \\ \dfrac{u_1 + u_2}{2} - u_4 & = 0 \end{cases}$$

   (b) ponctual connection for which, mesh II is the master mesh.

$$\begin{cases} u_1 - u_3 & = 0 \\ u_2 - u_5 & = 0 \end{cases}$$

   (c) connection at a mean sens allowing the transmition of constant forces.

$$\int_{-e}^{e} 1.(u^{II} - u^{I})dx = 0$$

   thus

$$\int_{-e}^{0} \frac{u_4 - u_5}{e} x dx + \int_{0}^{e} \frac{u_3 - u_4}{e} x dx + \int_{-e}^{e} \{u_4 - \frac{u_1 - u_2}{2} x - \frac{u_1 - u_2}{2e}\} dx = 0$$

   The condition on the degrees of freedom is then:

$$\frac{u_3 + u_5}{2} + u_4 - u_1 - u_2 = 0$$

   (d) connection at a mean sens allowing the transmission of linear forces: to the conditions obtained at the previous question we had:

$$\int_{-e}^{e} x.(u^{II} - u^{I})dx = 0$$

   thus

$$\int_{-e}^{0} \frac{u_4 - u_5}{e} x^2 dx + \int_{0}^{e} \frac{u_3 - u_4}{e} x^2 dx + \int_{-e}^{e} \{u_4 x - \frac{u_1 - u_2}{2} x^2 - \frac{u_1 - u_2}{2e} x\} dx = 0$$

   thus

$$u_3 - u_5 - u_1 + u_2 = 0$$

   The two conditions are then:

$$\begin{cases} \dfrac{u_3 + u_5}{2} + u_4 - u_1 - u_2 & = 0 \\ u_3 - u_5 - u_1 + u_2 & = 0 \end{cases}$$

A.3 Membrane problem

A.3.1 Text

We use a 1d approximation of a membrane submited to a tension T and to an extrenal pressure p on its lower surface. The two extremities are clamped in order that the vertical displacement is nul (figure A.3). $v(x)$ denotes the vertical displacement of the nodes of the membrane.

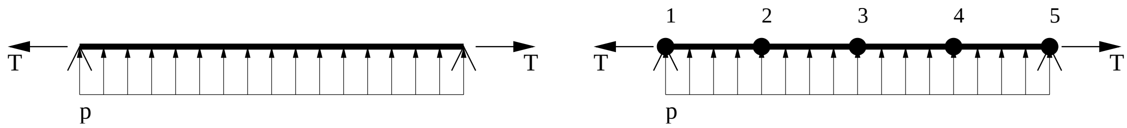

Figure A.3: Membrane under pressure and discretisation

Part 1 - Boundary conditions A finite element discretisation of the membrane is used. It presents four linear element which length is e (figure A.3 right).

v_i denotes the value of the field $v(x)$ on node i. As the ellemnt all have the same length, the elementary stiffness matrix and the elementary generalized force vector, corresponding to a distributed pressure p, are:

$$[K_{el}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \qquad ; \qquad \{F_{el}\} = \left\{ \begin{array}{c} \frac{pe}{2} \\ \frac{pe}{2} \end{array} \right\}$$

where $k = \frac{T}{e}$ is the membrane stiffness. In the following, the value k is kept in the expression of the stiffness matrices.

1. Assemble the global stiffness matrix and the generalized force vector.
2. Give the matrix form of the boundary condition.
3. Built the system that have to be solved when those conditions are taken into account using the *substitution* technique.
4. Solve the problem and draw the shape of the membrane.

In the following, we'll keep in assembled matrix in which we have eliminate the prescribed degrees of freedom.

Part 2 : frictionless unilateral contact with gap The displacement of the membrane is now limitated by the presence of a rigid body (figure A.4). j is The distance between the membrane and the rigid body.

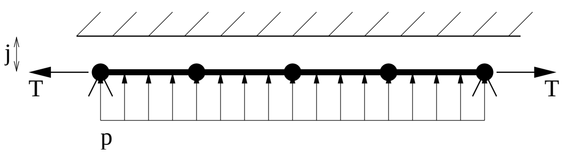

Figure A.4: Frictionless contact with gap

1. Write the local contact conditions between the membrane and the rigi body.

2. Give the matrix form of the discrete displacement condition. Is there an equivalence between the discrete condition and the continuous one?

3. Solve the problem using the status method in the case where $j = \frac{3pe}{2k}$. For that, on shall take the conditions into account using the Lagrange multiplier method in which λ_i will denote the multiplier associated to node i. Draw the deformed shape.

   Note : we shall recall that the Lagrange multiplier is the oppsite of the reaction force on the contact zone.

Contact with incompatible meshes The membrane can now come into contact with a deformable body whose width is e situated at a distance j (figure A.5). $v_m(x)$ denotes the vertical

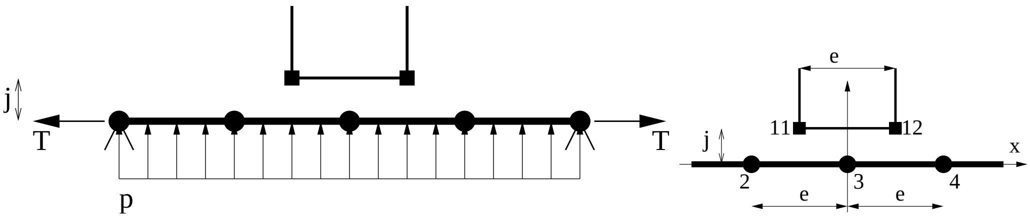

Figure A.5: Incompatible meshes

displacement of the membrane and $v_s(x)$ the one of the body.

We use a linear finite elemnt discretisation of the vertical displacement of the solid wich is not compatible with the one of the membrane (wich is the same than the one used in the other parts). Figure A.5 (right) gives the local parametrization.

Regarding the discretisations, the displacement fields can be expressed, on the contact zone, by:

$$v_s(x) = \frac{v_{12} - v_{11}}{e}\, x + \frac{v_{12} + v_{11}}{2}, \quad \forall x \in [-\frac{e}{2}, \frac{e}{2}] \quad ; \quad v_m(x) = \begin{cases} \dfrac{v_4 - v_3}{e} x + v_3 & , \quad \text{if } x \in [0, e] \\[2ex] \dfrac{v_3 - v_2}{e} x + v_3 & , \quad \text{if } x \in [-e, 0] \end{cases}$$

1. Write the continuous local frictionless contact conditions with gap between the deformable body and the membrane.

2. explain the notions of master and slave meshes for taking into account the nodale non penetration conditions.

3. Explain the notion of *connection at a mean sens*.

4. Give the conditions that have to be taken into account on the degrees of freedom depending on the type of connection:

   (a) ponctual connection for which mesh of the deformable solid is the master one.
   (b) ponctual connection for which mesh of the membrane is the master one.
   (c) connection at a mean sens allowing the transmission of constant forces.
   (d) connection at a mean sens allowing the transmission of linear forces.

5. We now consider that the body is rigid and fixed.

   (a) Give the discrete condition on the membraner when the conditions are taken into account using the technique proposed at 4(c).
   (b) Solve the problem using the status method.
   (c) Is the non penetration condition satisfied every where on the contact zone?

A.3.2 Correction

Part 1 - Boundary conditions

1. Stiffness matrix and generalized force vector:

$$[K] = \begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix} \quad ; \quad \{F\} = \begin{Bmatrix} \frac{pe}{2} \\ pe \\ pe \\ pe \\ \frac{pe}{2} \end{Bmatrix}$$

2. Matrix form of the conditions

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

3. system to be solved:

$$\begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} pe \\ pe \\ pe \end{Bmatrix}$$

4. Solution:

$$v_2 = \frac{3pe}{2k}; \qquad v_3 = \frac{2pe}{k}; \qquad v_4 = \frac{3pe}{2k}$$

Part 2 - frictionless unilateral contact with gap

1. Contact conditions: there is no friction and the interaction is inly in the vertical direction:

$$v(x) \leq j \quad ; \quad f(x) \leq 0 \quad ; \quad f(x).v(x) = 0$$

   where $f(x)$ is the force density of the body on the membrane.

2. Discrete form of the displacement condition:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \end{Bmatrix} \leq \begin{Bmatrix} j \\ j \\ j \end{Bmatrix}$$

3. Resolution using the status method:

   (a) The strict condition is prescribed:

$$\begin{bmatrix} 2k & -k & 0 & 1 & 0 & 0 \\ -k & 2k & -k & 0 & 1 & 0 \\ 0 & -k & 2k & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{Bmatrix} = \begin{Bmatrix} pe \\ pe \\ pe \\ j \\ j \\ j \end{Bmatrix}$$

   which solution is:

$$\left\{ \begin{array}{lll} \lambda_2 & = pe - kj = -\frac{pe}{2} & < 0 \\ \lambda_3 & = pe & > 0 \\ \lambda_4 & = pe - kj = -\frac{pe}{2} & < 0 \end{array} \right. \quad ; \quad \left\{ \begin{array}{lll} v_2 & = j & = \frac{3pe}{2} \\ v_3 & = j & = \frac{3pe}{2} \\ v_4 & = j & = \frac{3pe}{2} \end{array} \right.$$

Reaction forces on nodes 2 and 4 are positive so the conditions on these nodes have to be removed.

(b) Strict condition on node 3 is only prescribed:

$$\begin{bmatrix} 2k & -k & 0 & 0 \\ -k & 2k & -k & 1 \\ 0 & -k & 2k & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \\ \lambda_3 \end{Bmatrix} = \begin{Bmatrix} pe \\ pe \\ pe \\ j \end{Bmatrix}$$

the solution is then:

$$\lambda_2 = pe > 0 \qquad ; \qquad \begin{cases} v_2 & = \dfrac{pe}{2k} + kj & = \dfrac{5pe}{4k} \\ v_3 & = j & = \dfrac{3pe}{2k} \\ v_4 & = \dfrac{pe}{2k} + kj & = \dfrac{5pe}{4k} \end{cases}$$

Reaction force on node 3 is negative, the displacement conditions are satisfied.

Part3 - Contact with incompatible meshes

1. Frictionless contact conditions:

$$u_m(x) - u_s(x) \leq j$$

2. Master and slave meshes (see lecture notes)
3. Connection at a mean sens (see lecture notes)
4. Discrete condition depending on the type of connection:

(a) ponctual connection for which mesh of the deformable solid is the master one:

$$u_3 - \frac{u_{12} + u_{11}}{2} \leq j$$

(b) ponctual connection for which mesh of the membrane is the master one:

$$\begin{cases} \dfrac{u_2 + u_3}{2} - u_{11} \leq j \\ \dfrac{u_4 + u_3}{2} - u_{12} \leq j \end{cases}$$

(c) connection at a mean sens allowing the transmission of constant forces:

$$\int_{-e/2}^{e/2} (u_m(x) - u_s(x) - j)dx \leq 0$$

thus, after integration:

$$\frac{1}{8}v_2 + \frac{1}{8}v_4 + \frac{3}{4}v_3 - \frac{1}{2}v_{12} - \frac{1}{8}v_{11} \leq j$$

(d) connection at a mean sens allowing the transmission of linear forces. The following condition have to be added to the previous one:

$$\int_{-e/2}^{e/2} x(u_m(x) - u_s(x) - j)dx \leq 0$$

that is, after integration:

$$v_4 - v_2 - 2v_{12} + 2v_{11} \leq 0$$

5. We now consider that the body is rigid and fixed.

   (a) The contact condition becomes:

   $$\frac{1}{8}v_2 + \frac{1}{8}v_4 + \frac{3}{4}v_3 \leq j$$

   (b) Resolution by the status method:

   $$\begin{bmatrix} 2k & -k & 0 & \frac{1}{8} \\ -k & 2k & -k & \frac{3}{4} \\ 0 & -k & 2k & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 \end{bmatrix} \left\{ \begin{array}{c} v_2 \\ v_3 \\ v_4 \\ \lambda \end{array} \right\} = \left\{ \begin{array}{c} pe \\ pe \\ pe \\ j \end{array} \right\}$$

   what gives:

   $$\lambda = \frac{5}{2}pe - \frac{4}{3}kj$$

   using the value of j proposed in section 2, one have:

   $$\lambda = \frac{1}{2}pe > 0$$

   The reaction force is negative, the condition is then satisfied. The solution is then:

   $$\lambda = \frac{1}{2}pe \qquad ; \qquad \left\{ \begin{array}{ll} v_2 & = \frac{5}{4}\frac{pe}{k} \\ v_3 & = \frac{25}{16}\frac{pe}{k} \\ v_2 & = \frac{5}{4}\frac{pe}{k} \end{array} \right.$$

   (c) One have $v_3 > j$ thus the non penetration condition is not satisfied locally on node 3.

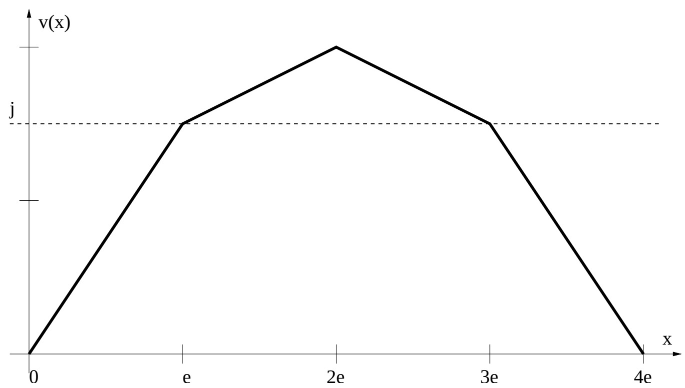

Figure A.6: Solution of part 1

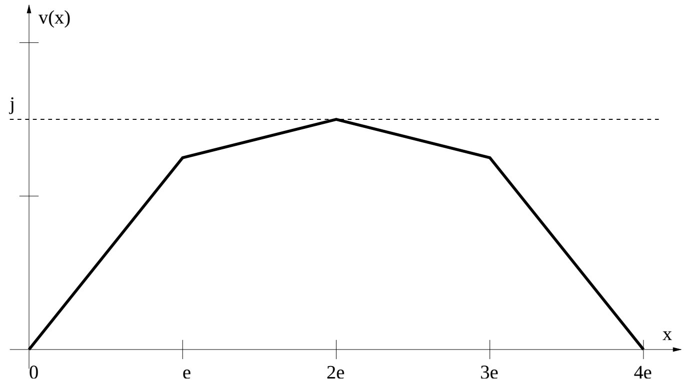

Figure A.7: Solution of part 2

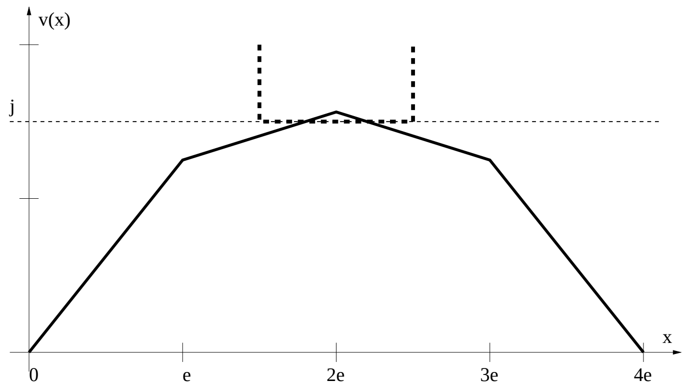

Figure A.8: Solution of part 3

A.4 Beam problem

A.4.1 Text

We consider a beam which length is $4L$, cantilevered at both ends and submitted to a bending force F in his middle (see figure A.9). Because of the symmetry of the problem, we only study one half of the beam by priscribing a zero rotation condition in the middle. $v(x)$ denotes the vertical displacement of the points of the beam and the Euler-Bernoulli approximation is considered. A

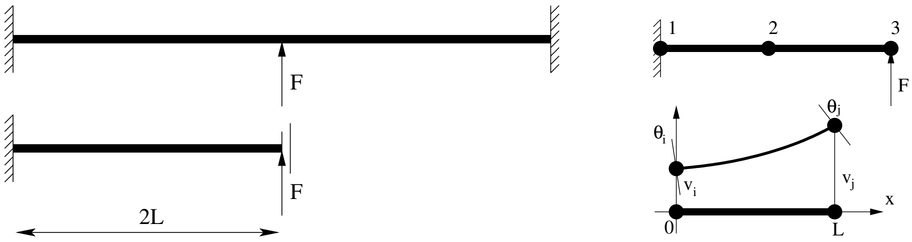

Figure A.9: Beam, symmetry and discretisation

finite element discretisation is used on this half-beam. It contains two elements with a L length (figure A.9 right). v_i denotes the value of the displacement field $v(x)$ on node i. θ_i denotes the section rotation on node i. Using the Euler-Bernoulli approximation, the displacement field $v(x)$ on such an element is:

$$v(x) = \phi_i(x)v_i + \varphi_i(x)\theta_i + \phi_j(x)v_j + \varphi_j(x)\theta_j$$

with

$$\phi_i(x) = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}, \quad \varphi_i(x) = x - \frac{2x^2}{L} + \frac{x^3}{L^2}, \quad \phi_j(x) = \frac{3x^2}{L^2} - \frac{2x^3}{L^3}, \quad \varphi_j(x) = \frac{x^2}{L} + \frac{x^3}{L^2}$$

Using this discretisation, as the element all have the same length L, the elementary stiffness matrix on one element is:

$$[K_{el}] = a \begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix} \text{ avec } a = \frac{2EI}{L^3} \quad ; \quad \text{associated dof: } \left\{ \begin{array}{c} v_i \\ \theta_i \\ v_j \\ \theta_j \end{array} \right\}$$

where E is the Young's modulus of the material, I the bending inertia of the section and L the length of an element.

Part 1 - Boundary conditions

1. Assemble the global stiffness matrix and the generalized force vector.
2. Give the matrix form of the boundary condition.
3. Built the system that have to be solved when those conditions are taken into account using the *substitution* technique.
4. Solve the problem and draw the shape of the beam.
5. show that, when the system is condensed on the displacement degrees of freedom v_2 and v_3, it becomes:

$$\begin{bmatrix} 12a & -6a \\ -6a & \frac{15a}{4} \end{bmatrix} \left\{ \begin{array}{c} v_2 \\ v_3 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ F \end{array} \right\} \quad \text{where } v_1 = \theta_1 = \theta_3 = 0 \quad \text{and} \quad \theta_2 = \frac{3v_3}{4L}$$

This last system is the one used in the following.

Part 2 : frictionless unilateral contact with gap The displacement of the beam is now limitated by the presence of a rigid body (figure A.10). j is The distance between the membrane and the rigid body.

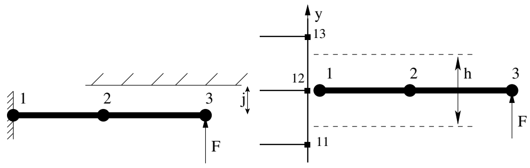

Figure A.10: Frictionless contact with gap and model connection

1. Write the local contact conditions between the beam and the rigi body.
2. Give the matrix form of the discrete displacement condition. Explain why this not equivalence between the discrete form and the continuous form of this condition.
3. Solve the problem using the status method in the case where $j = \frac{F}{2a}$. For that, on shall take the conditions into account using the Lagrange multiplier method in which λ_i will denote the multiplier associated to displacement v_i. Draw the deformed shape of the beam.

   Note : we shall recall that the Lagrange multiplier is the opposite of the reaction force on the contact zone.
4. Is the displacement condition satisfied at each point of element $2-3$?
5. We now want to prescribed the non penetration condition as a mean condition on element $2-3$. Write the condition on the displacement and rotation degrees of freedom corresponding to a constant force mean connection.

Incompatible model connection The left cantilevered side is now replaced by a connection with deformable media (figure A.10 right). $u_m(x)$ and $v_m(x)$ denotes the horizontal and vetical components of the deformable media. A linear finite element discretisation of the media is used: the node located on the connection are indicated on the figure. They have a uniform size e. In the connection region, with such a discretisation, the considered fields are locally expressed:

$$u_m(x) = \begin{cases} \dfrac{u_{13}-u_{12}}{e}y + u_{12} \quad , & \text{if } y \in [0,e] \\[2ex] \dfrac{u_{12}-u_{11}}{e}y + u_{12} \quad , & \text{if } y \in [-e,0] \end{cases} \; ; \; v_m(x) = \begin{cases} \dfrac{v_{13}-v_{12}}{e}y + v_{12} \quad , & \text{if } y \in [0,e] \\[2ex] \dfrac{v_{12}-v_{11}}{e}y + v_{12} \quad , & \text{if } y \in [-e,0] \end{cases}$$

1. Write the ponctual connection conditions between the degrees of freedom of the media and the ones of the node 1 of the beam when the rotation of the media is expressed from the displacement of nodes 11 and 13.
2. Write the mean connection conditions that are to be prescribed on the degrees of freedom for the transmission of forces and moment knowing that the real thickness of the beam is h.

A.4.2 Correction

Part 1 - Boundary conditions

1. Stiffness matrix and generalized force vector. The system is:

$$a\begin{bmatrix} 6 & 3L & -6 & 3L & 0 & 0 \\ 3L & 2L^2 & -3L & L^2 & 0 & 0 \\ -6 & -3L & 12 & 0 & -6 & 3L \\ 3L & L^2 & 0 & 4L^2 & -3L & L^2 \\ 0 & 0 & -6 & -3L & 6 & -3L \\ 0 & 0 & 3L & L^2 & -3L & 2L^2 \end{bmatrix}\left\{\begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{matrix}\right\} = \left\{\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ F \\ 0 \end{matrix}\right\}$$

2. Matrix form of the conditions

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\left\{\begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{matrix}\right\} = \left\{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\}$$

3. system to be solved:

$$\begin{bmatrix} 12 & 0 & -6 \\ 0 & 4L^2 & -3L \\ -6 & -3L & 6 \end{bmatrix}\left\{\begin{matrix} v_2 \\ \theta_2 \\ v_3 \end{matrix}\right\} = \left\{\begin{matrix} 0 \\ 0 \\ F \end{matrix}\right\}$$

4. Solution:

$$v_2 = \frac{2F}{3a}; \qquad \theta_2 = \frac{F}{aL}; \qquad v_3 = \frac{4F}{3a}$$

5. Condensing the system on dof v_2 and v_3, we obtain:

$$\begin{bmatrix} 12a & -6a \\ -6a & \frac{15a}{4} \end{bmatrix}\left\{\begin{matrix} v_2 \\ v_3 \end{matrix}\right\} = \left\{\begin{matrix} 0 \\ F \end{matrix}\right\} \quad \text{and a post-computation gives} \quad \theta_2 = \frac{3v_3}{4L}$$

Part 2 - frictionless unilateral contact with gap

1. Contact conditions: there is no friction and the interaction is only in the vertical direction:

$$v(x) \leq j \quad ; \quad f(x) \leq 0 \quad ; \quad f(x).(v(x) - j) = 0$$

   where $f(x)$ is the force density of the body on the membrane.

2. Discrete form of the displacement condition:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\left\{\begin{matrix} v_2 \\ v_3 \end{matrix}\right\} \leq \left\{\begin{matrix} j \\ j \end{matrix}\right\}$$

3. Resolution using the status method:

   (a) The strict condition is prescribed:

$$\begin{bmatrix} 12a & -6a & 1 & 0 \\ -6a & \frac{15a}{4} & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\left\{\begin{matrix} v_2 \\ v_3 \\ \lambda_2 \\ \lambda_3 \end{matrix}\right\} = \left\{\begin{matrix} 0 \\ F \\ j \\ j \end{matrix}\right\}$$

which solution is:

$$\left\{\begin{array}{llll}\lambda_2 & =-6aj & =-3F & <0 \\ \lambda_3 & =F+\frac{9}{4}aj & =\frac{17F}{8} & >0\end{array}\right. \quad ; \quad \left\{\begin{array}{lll}v_2 & =j & =\frac{F}{2a} \\ v_3 & =j & =\frac{F}{2a}\end{array}\right.$$

Reaction forces on nodes 2 is positive so the condition on this node have to be removed.

(b) Strict condition on node 3 is only prescribed:

$$\begin{bmatrix} 12a & -6a & 0 \\ -6a & \frac{15a}{4} & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \\ \lambda_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \\ j \end{Bmatrix}$$

the solution is then:

$$\lambda_3 = \frac{5}{8}F > 0 \quad ; \quad \left\{\begin{array}{llll} v_2 & =\dfrac{j}{2} & =\dfrac{F}{4a} & \leq j \\ v_3 & =j & =\dfrac{F}{2a} & \leq j \end{array}\right.$$

Reaction force on node 3 is negative, the displacement conditions are satisfied.

(c) The condition is satisfied on each point

(d) Mean condition on element $2-3$:

$$\int_0^L (v(x)-j)dx \leq 0$$

that gives after intgration, using the basis function and the fact that $\theta_3 = 0$:

$$\frac{L}{2}v_2 - \frac{L^2}{4}\theta_2 + \frac{L}{2}v_3 - Lj \leq 0$$

so the condition to be prescribed is:

$$\frac{1}{2}v_2 - \frac{L}{4}\theta_2 + \frac{1}{2}v_3 - j \leq 0$$

Part 3 - Contact with incompatible meshes

1. The ponctual conditions are:

$$u_1 = u_{12} \quad ; \quad v_1 = v_{12} \quad ,; \quad \theta_1 = \frac{u_{11}-u_{13}}{2e}$$

2. The mean conditions that allow the transmission of forces and moment are:

$$\int_{-h/2}^{h/2} \underline{F}_i^*(\underline{u}_m(y) - \underline{u}_p(y))dy = 0, \forall i = 1,2,3$$

where:

$$\underline{F}_1^* = \underline{x} \quad ; \quad \underline{F}_2^* = \underline{y} \quad ; \quad \underline{F}_3^* = y\underline{x}$$

what gives:

$$\left\{\begin{array}{l} \displaystyle\int_{-h/2}^{h/2}(u_m(y)-u_p(y))dy = 0 \\ \displaystyle\int_{-h/2}^{h/2}(v_m(y)-v_p(y))dy = 0 \\ \displaystyle\int_{-h/2}^{h/2}y(u_m(y)-u_p(y))dy = 0 \end{array}\right.$$

with, for the beam displacement:

$$u_p(y) = u_1 - \theta_1 y \quad ; \quad v_p(y) = v_1$$

what gives:

$$\begin{cases} \displaystyle\int_{-h/2}^{h/2} u_m(y)dy - hu_1 & = 0 \\ \displaystyle\int_{-h/2}^{h/2} v_m(y)dy - hv_1 & = 0 \\ \displaystyle\int_{-h/2}^{h/2} yu_m(y)dy + \frac{h^3}{12}\theta_1 & = 0 \end{cases}$$

Using the expression of the displacement of the media, one obtains:

$$\begin{cases} u_1 & = \dfrac{h}{8e}(u_{11} + u_{13} - 2u_{12}) + 2u_{12} \\ v_1 & = \dfrac{h}{8e}(v_{11} + v_{13} - 2v_{12}) + 2v_{12} \\ \theta_1 & = \dfrac{u_{11} - u_{13}}{2e} \end{cases}$$

which are the conditions that are to be prescribed.