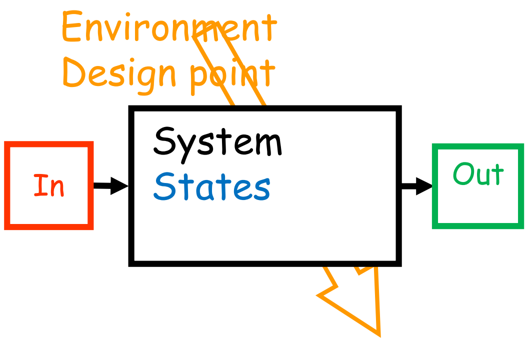
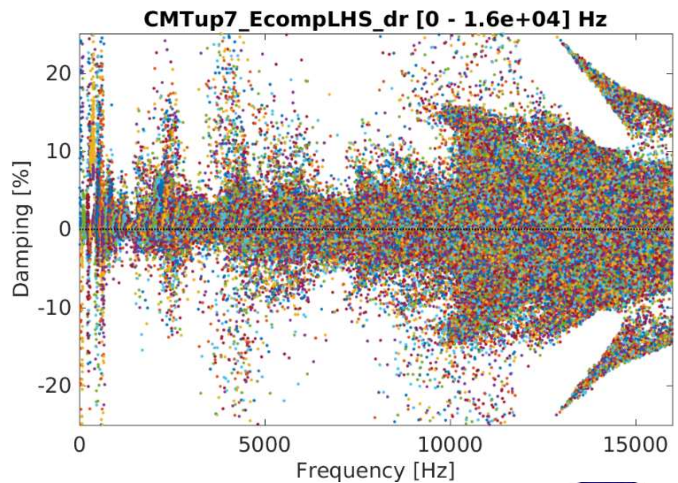
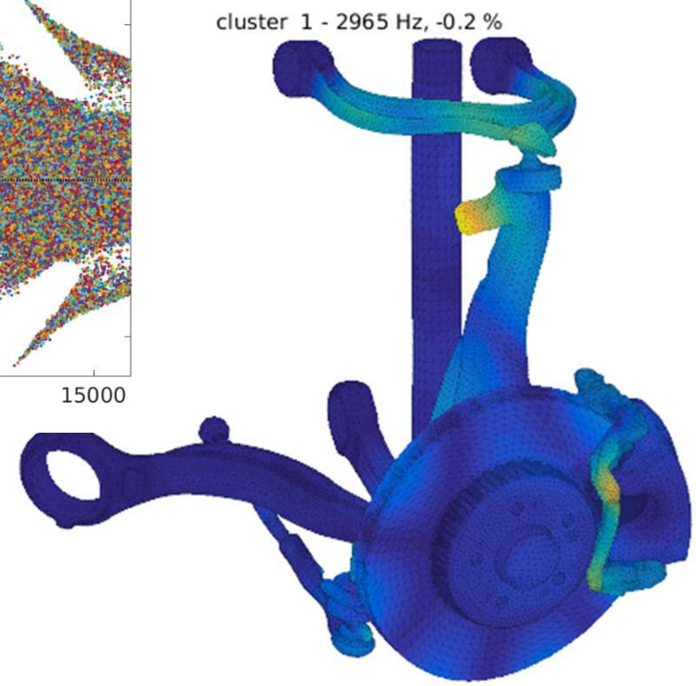
Course 7-8 parametric problems







- Parameters
- Sensitivity
- Reanalysis
- Updating

SDTools VIBRATION SOFTWARE & CONSULTING

Arts et Métiers Institute of Technology

MS2SC http://savoir.ensam.eu/moodle/course/view.php?id=1874

PROVIR http://savoir.ensam.eu/moodle/course/view.php?id=490

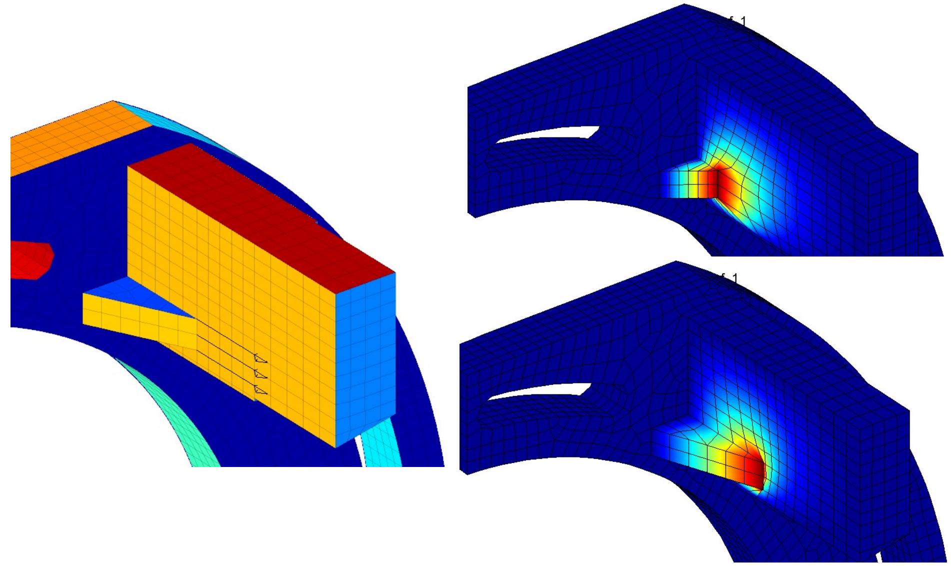
Geometry parametrization / morphing

- Shape optimization / morphing

$$P(x) = \sum_i \{p_i(x)\} q_{imaster}$$

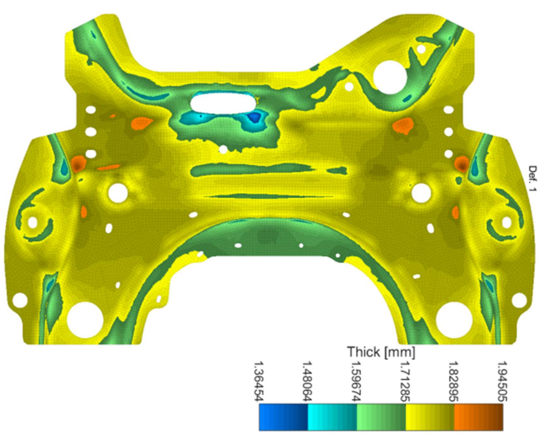
fe_shapeoptim BuildFromSel

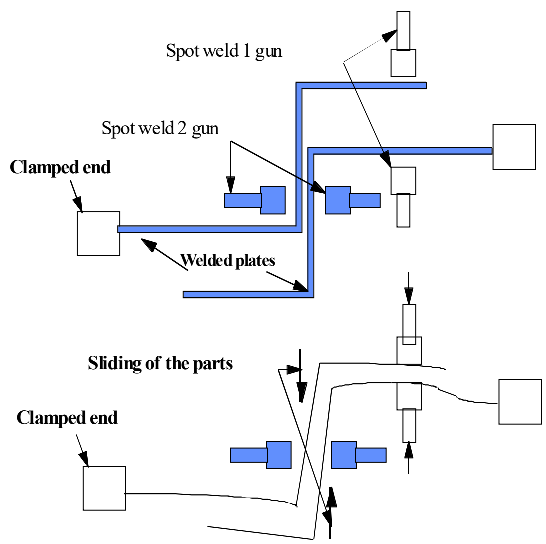
1) fix bottom face, 2) prescribe edge motion
3) deform edges (straight)
4) deform faces (flat)
5) deform interior (good elements ?)

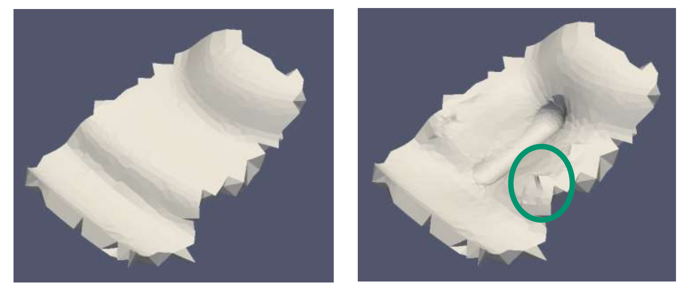


- Process simulation + field projection

fe_shapeoptim Interp





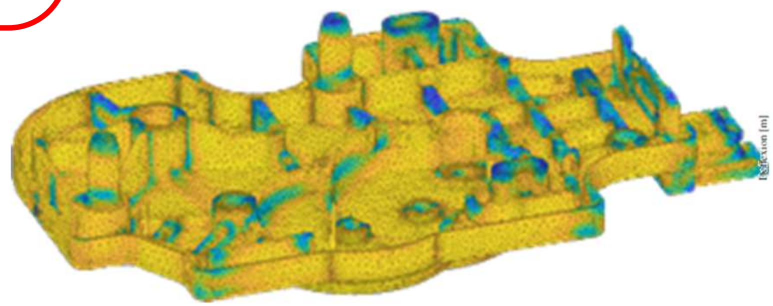


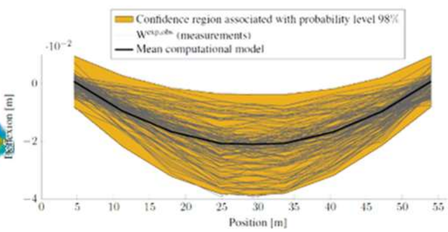
[1] de Paula, Rejdych, Chancelier, Vermot, Balmes, « On the influence of geometry updating on modal correlation of brake components. », in *Vibrations, Chocs & Bruit*, 2012.
[2] G. Vermot Des Roches, E. Balmes, et S. Nacivet, « Error localization and updating of junction properties for an engine cradle model », in ISMA, Leuven, Belgium, 2016, p. ID 372.
[3] E. Blain, « Etudes expérimentales et numériques de la dispersion vibratoire d'assemblages soudés par points », Ph.D. thesis, Ecole Centrale de Paris, 2000.

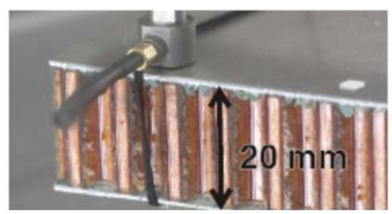
Direct problems : material parameters

- Uniform
- Field
- Equivalent
  (at certain scales)

- Geometry
- Material parameters
- Junction representation
- Equivalent parameters







- Basic parametrization tool : dependence on constitutive parameters C_{ij}

$$K = \int_\Omega \{\epsilon\}^T [C_{ij}] \{\epsilon\} = \sum_g B^T [C_{ij}] B\, w_g = \sum_{ij} C_{ij} \left[\sum_g B^T [C^u_{ij}] B w_g \right] = \sum_{ij} C_{ij} [K_{ij}]$$

Constitutive law parameterization

- Example : bar stiffness $K = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ proportional to E

- Classical problem shell thickness parameterized with

$$\beta_1 = t, \beta_2 = t^3, \beta_3 = t^2$$

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix}$$

- Current practice : weighted element sum

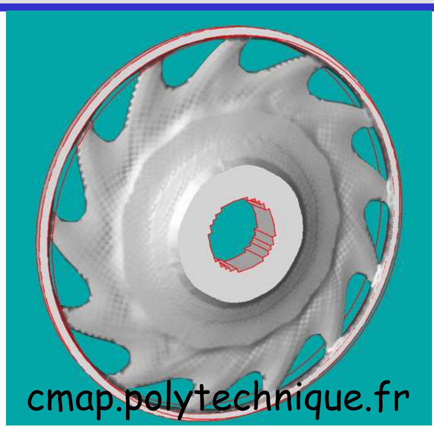
$$[M(p)] = \sum_{j=1}^{NE} \alpha_k(p)\,[M_k^e] \qquad [K(p)] = \sum_{j=1}^{NE} \beta_k(p)\,[K_k^e]$$

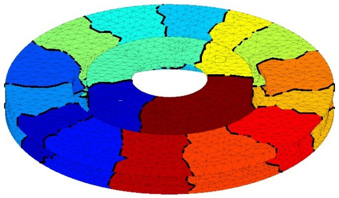
Element/model weights

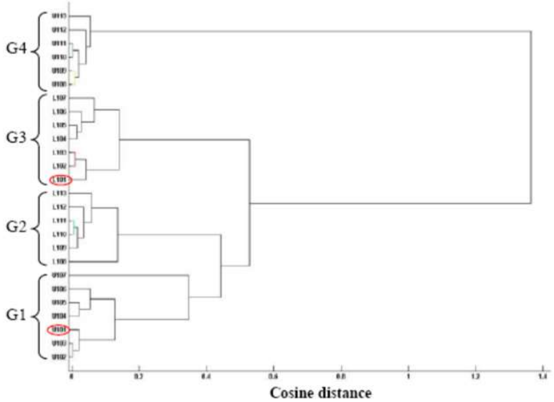
$$K(p) = \sum_e \alpha_e(p)[K^e]$$

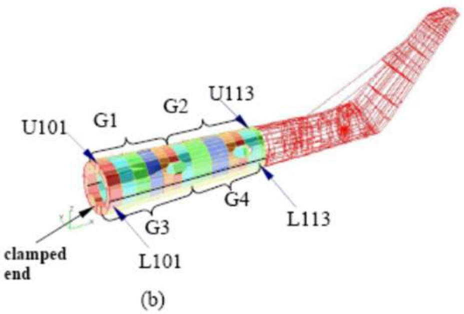
Weighted element matrices = standard

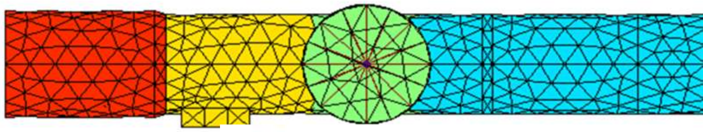
- Element-wise (topology optimization)
- Field/groupwise
  - parameter groups, …
  - solution of eigenvalue problem (polynomial chaos, Ghanem, Soize, …)
  - clustering (for example k-means a usual data-mining algorithm)

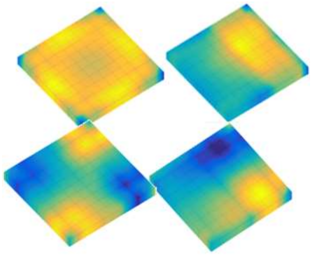


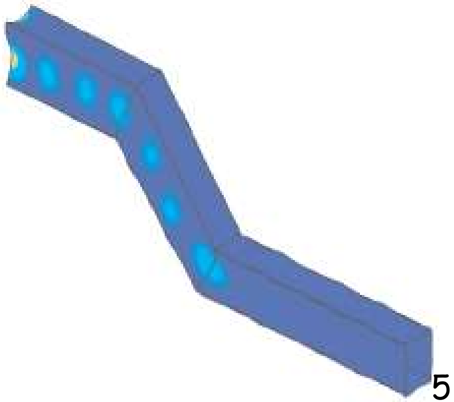












Clustering of Parameter Sensitivities: Examples from a Helicopter Airframe Model Updating Exercise. Shahverdi, Mottershead & All

SDT implementations : upcom / zCoef / stressCut

- Element wise $K(p) = \sum_e \alpha_k^e [K^e]$ www.sdtools.com/help/upcom.html

$$\text{mind} = \begin{bmatrix} M_s & M_e & K_s & K_e & \alpha_m & \alpha_k \\ \vdots & & & & & \\ elt & & & & & \end{bmatrix}$$

- Group wise $K(p) = \sum_g \alpha_g \; [K^g]$ www.sdtools.com/help/zCoef.html

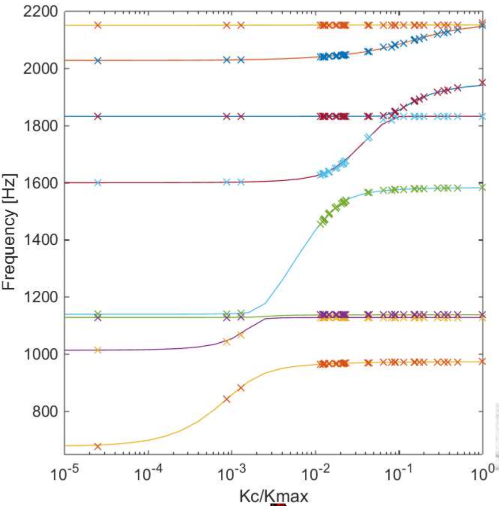
```
zCoef={'Klab','mCoef','zCoef0','zCoefFcn';
       'M'    1        0        '-w.^2';
       'Ke'   0        1        1+i*fe_def('DefEta',[]);
       'Kv'   0        1        'par(1)'};
```

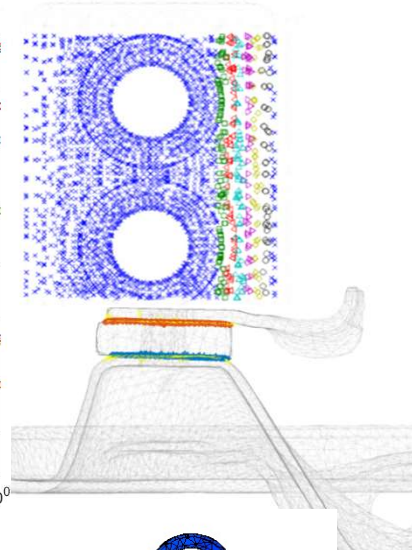
- Disassembly www.sdtools.com/help/corstress.html

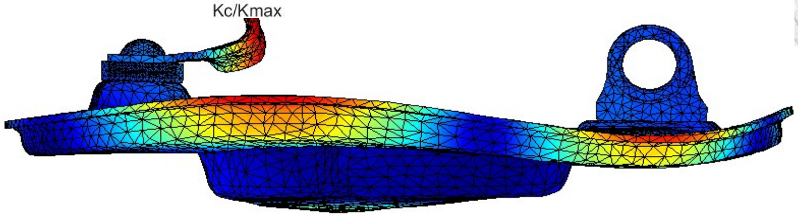
$$K(p) = [b][C_{constit}][c] = \begin{bmatrix} \ddots & & \\ & w_g J_g N_i^g & \\ & & \ddots \end{bmatrix}^T \begin{bmatrix} \ddots & & \\ & C_{ij}^g & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & \\ & N_i^g & \\ & & \ddots \end{bmatrix}_{(Ng \times Nstrain) \times N}$$

Parametrization of contact/sliding

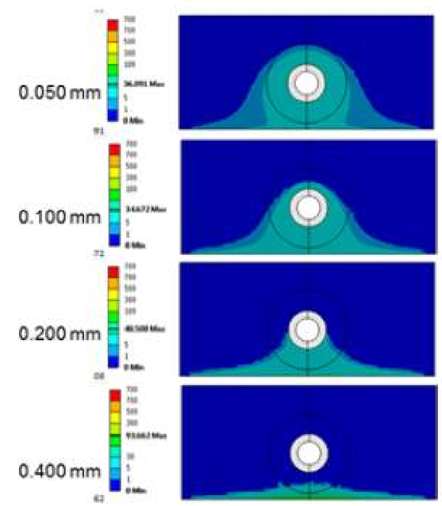
- Variable contact surface, contact, sliding



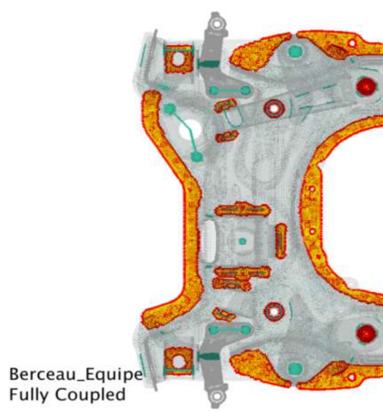


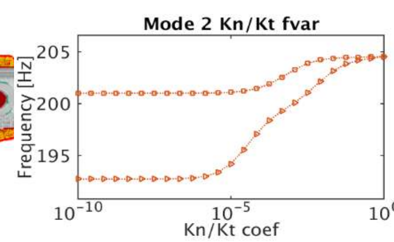


Chassis Brakes International
Eurobrake 2014



Goth, ISMA 2016





Balmes, ISMA 2016

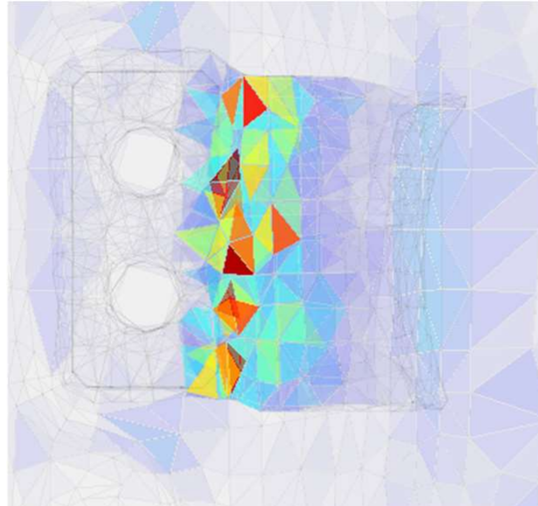
Sensitivity

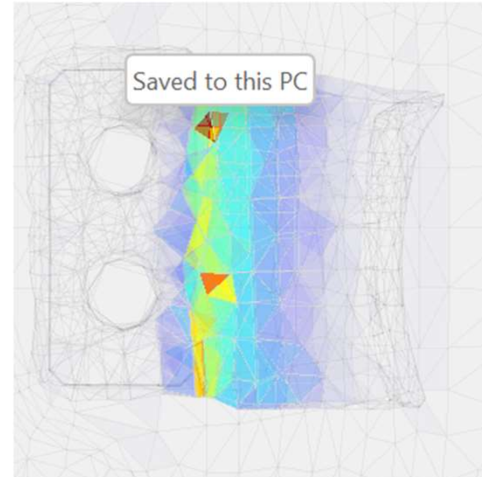
- Static direct and adjunct (poly section 10.2)
- Frequency and shape sensitivity (poly section 10.3)
- Exact and Fox-Kapoor (change to modal coordinates)

Tricks with energy display

$$\frac{\partial \omega_j^2}{\partial p} = \{\phi_j\}^T \left[\frac{\partial K}{\partial p} - \omega_j^2 \frac{\partial M}{\partial p}\right] \{\phi_j\}$$

$$\omega_j = \sqrt{\frac{\sum_e \; q^T K^e q}{\sum_e \; q^T M^e q}}$$

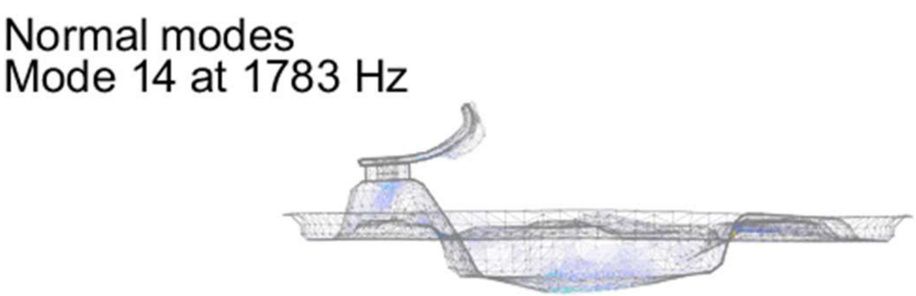


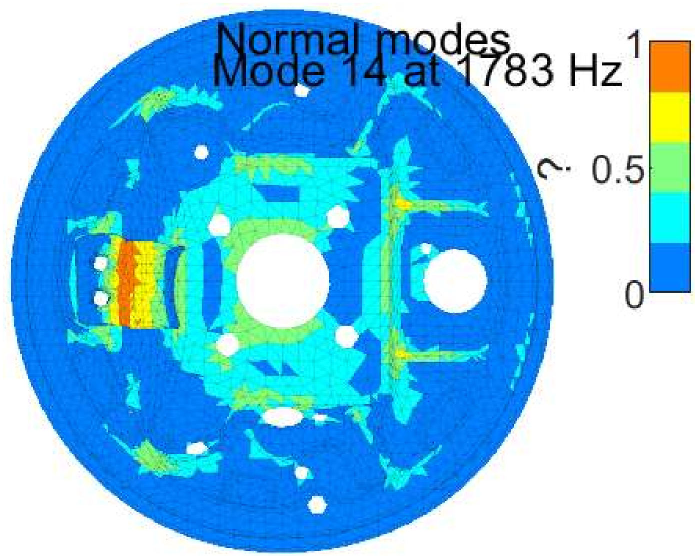


Views by

- Element energy (Abaqus ELSE)
- Energy density (ESEDEN)
- Energy in group

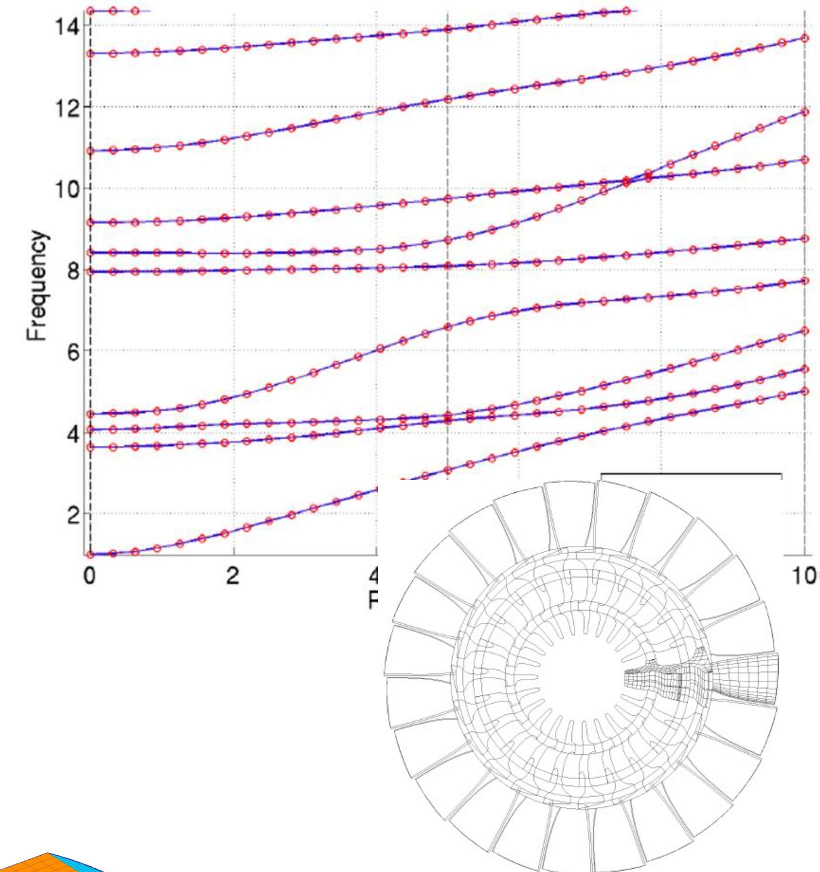
Give different perspectives

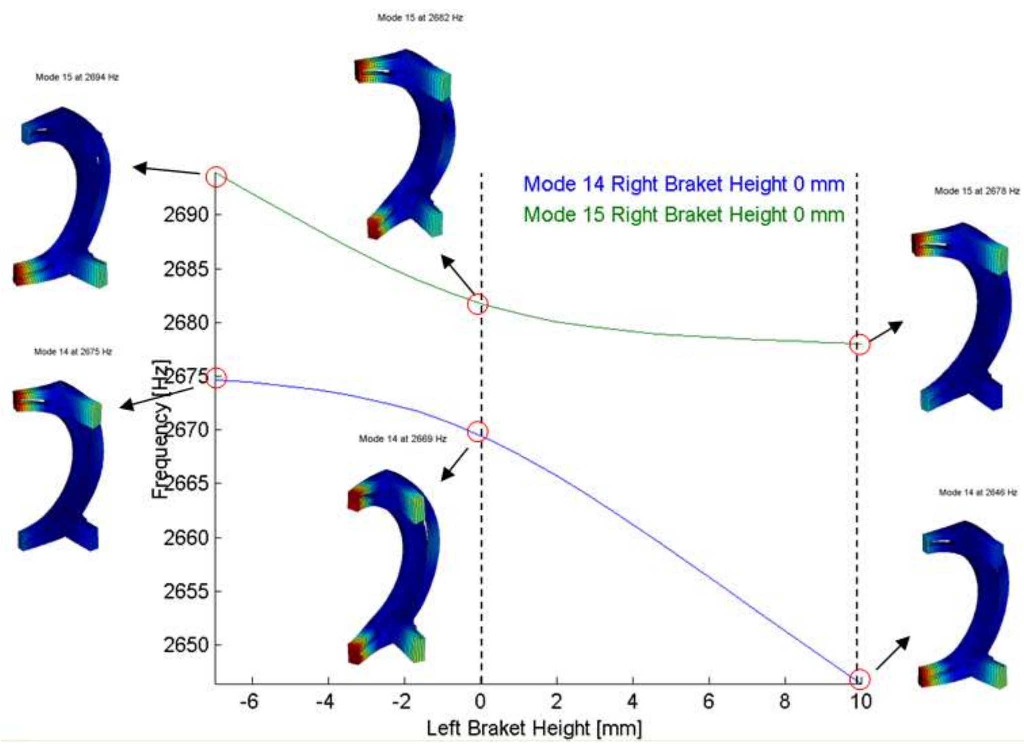


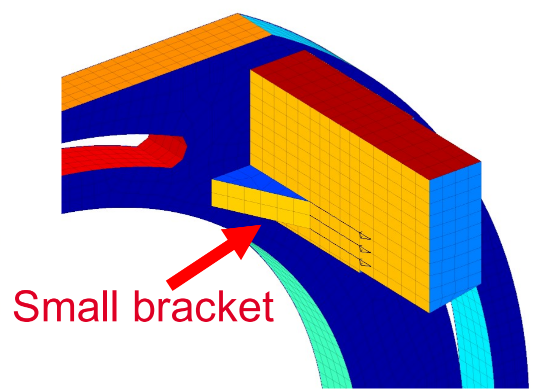


Mode crossing

- High sensitivity for close modes associated with mode crossing

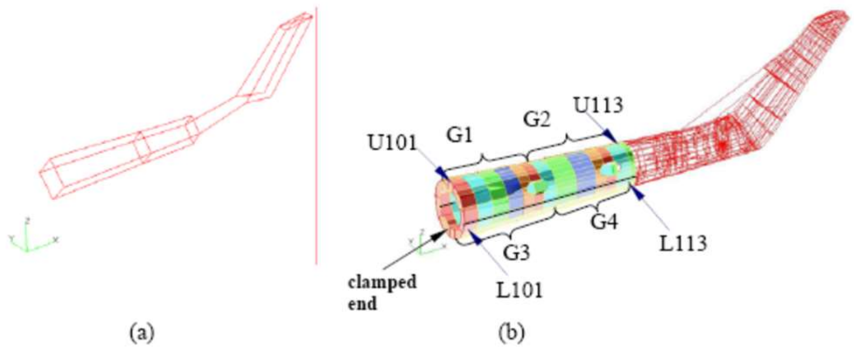


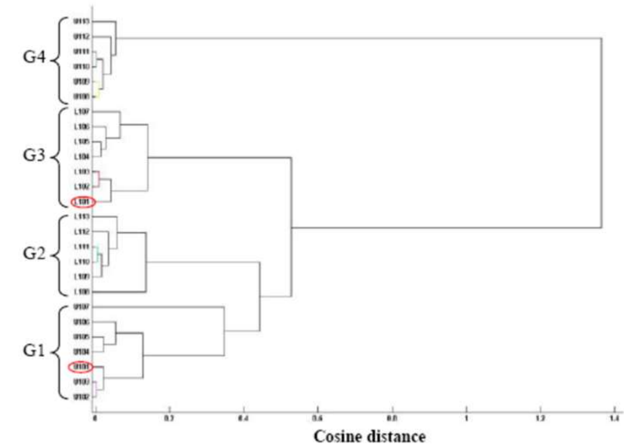




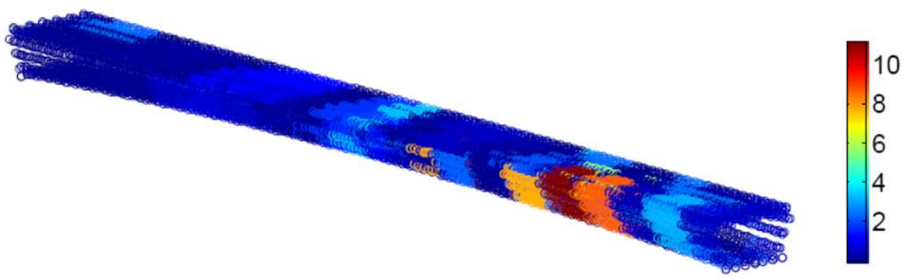
Clustering examples

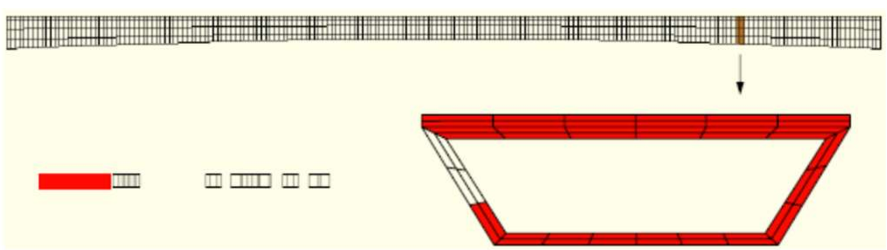
- Helicopter frame





- Bridge deck



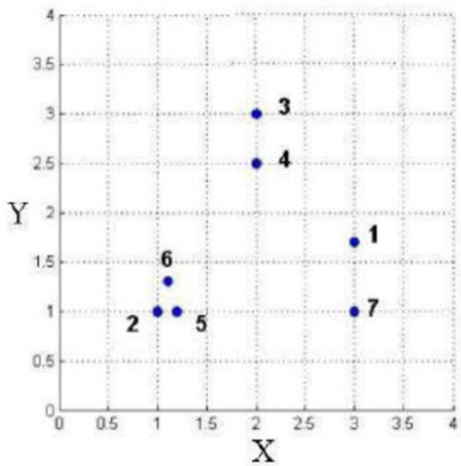


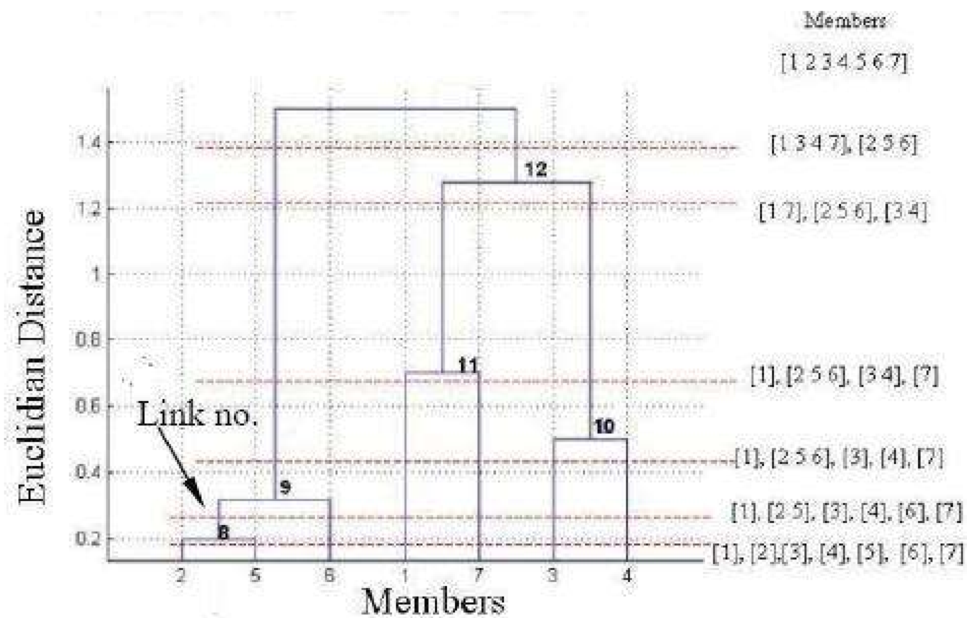
Clustering of Parameter Sensitivities: Examples from a Helicopter Airframe Model Updating Exercise.
Shahverdi, Mottershead & All
Statistical Model-Based Damage Localization: a Combined Subspace-Based and Substructuring Approach
Balmes, Basseville & All

Sensitivity / clustering

- Clustering techniques (k-means for example) can be used to group elements with similar effects
- Key mathematical notion : cosine distance (subspace in MATLAB)

$$d_{ij} = 1 - \sum_{k=1}^{p} x_{ik}\, x_{jk} \Big/ \left(\sum_{k=1}^{p} x_{ik}^{2} \sum_{k=1}^{p} x_{jk}^{2} \right)^{1/2}$$



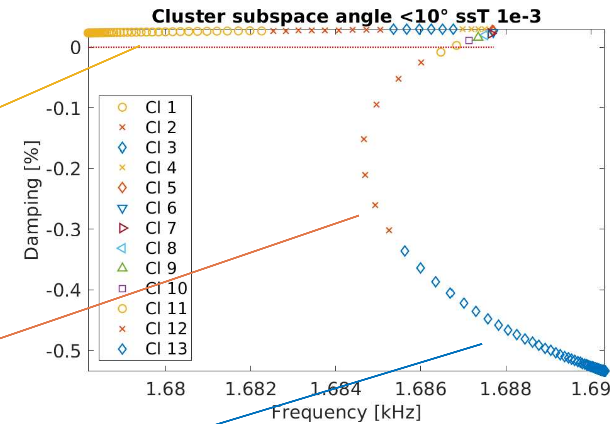


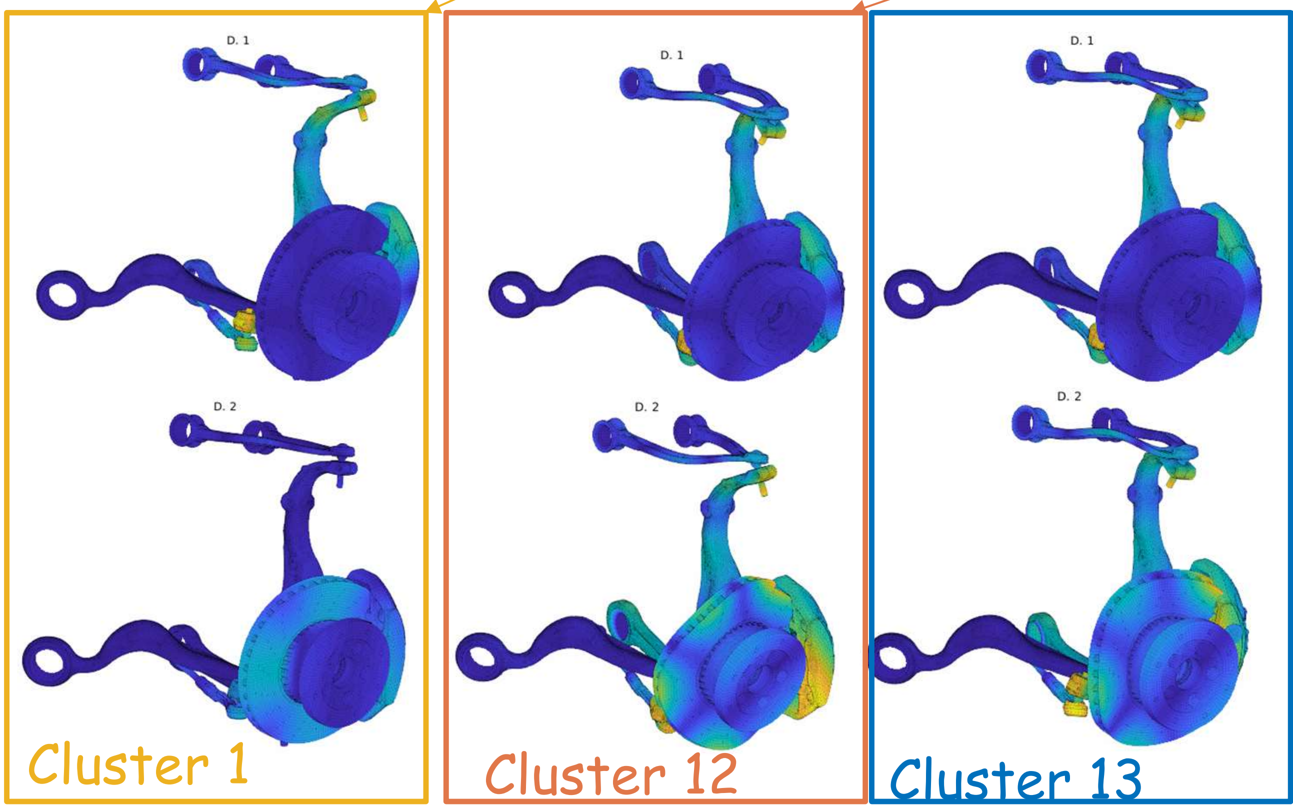
Link no.	linked objects		distance
8	2	5	0.2
9	6	8	0.32
10	3	4	0.5
11	1	7	0.7
12	10	11	1.28
	9	12	1.5

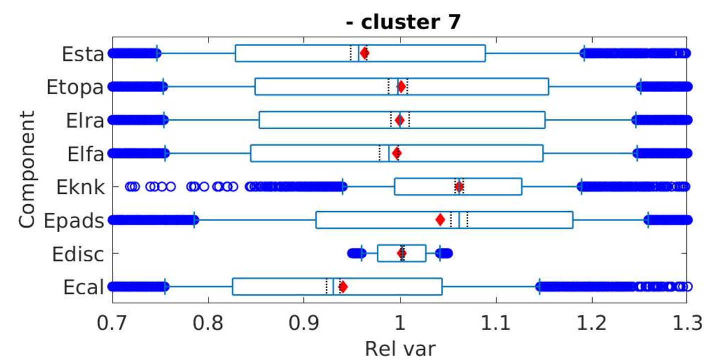
Clustering of Parameter Sensitivities: Examples from a Helicopter Airframe Model Updating Exercise. Shahverdi, Mottershead & All

Subspace clustering example

- Clustering
  - 2D complex mode basis
  - Subspace angle tolerance
- Parameter variations per cluster



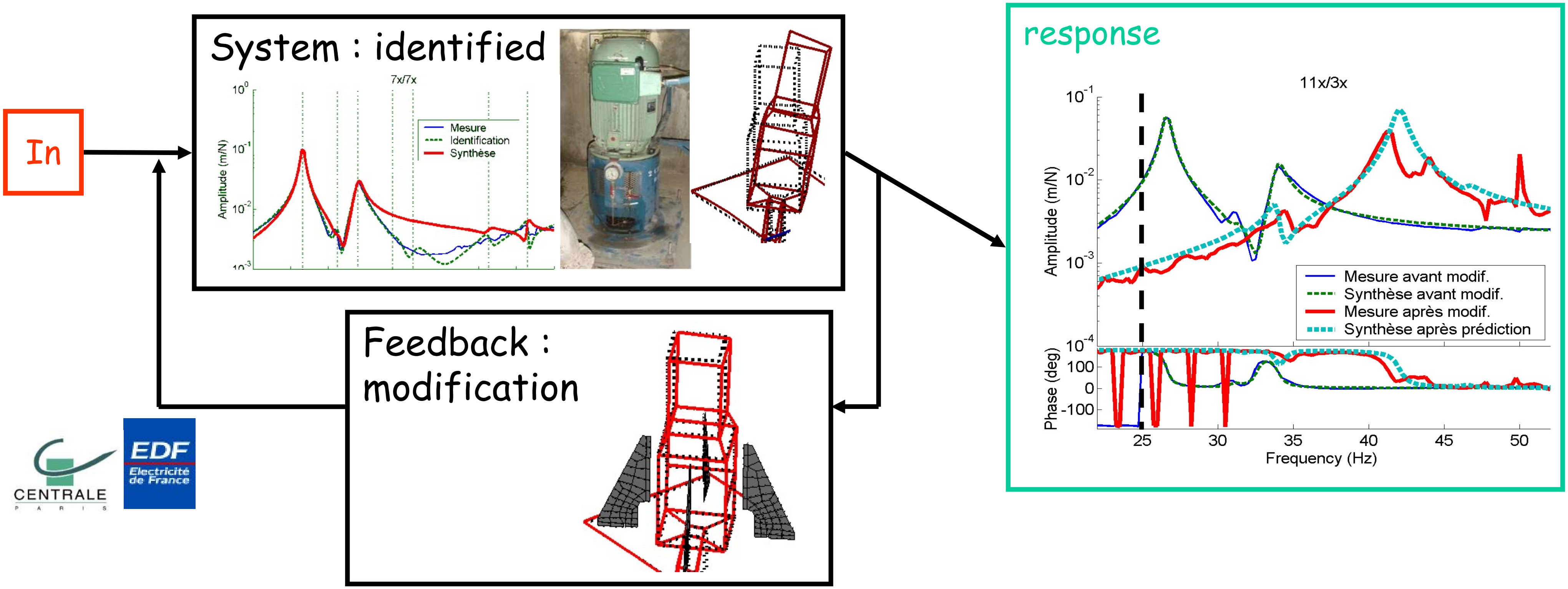




Topic 2 Reanalysis

- Rather than sensitivity use “reanalysis”

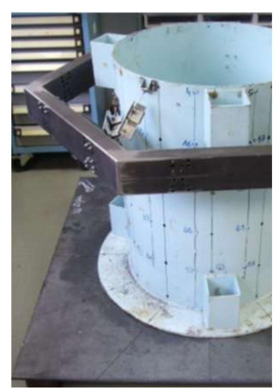
Reanalysis : model 1 config -> prediction other

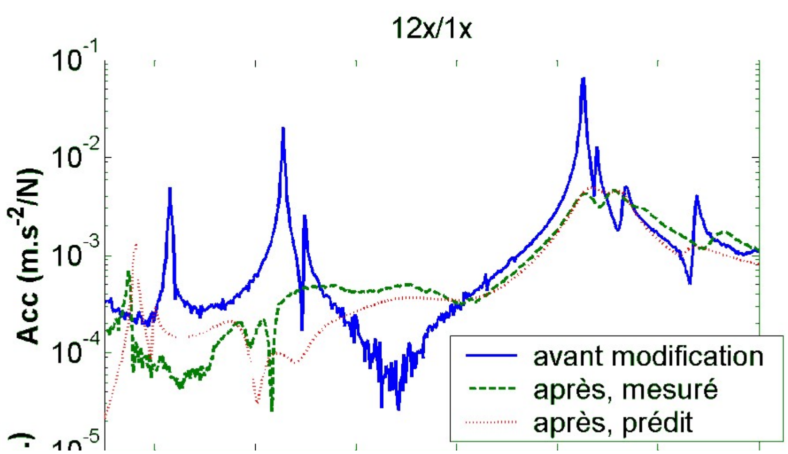


Test : example

SDM : structural dynamics modification

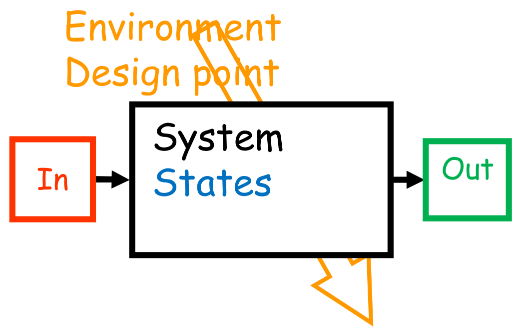
- distributed mass, stiffness or damping modifications
- Restricted to low frequency global modes

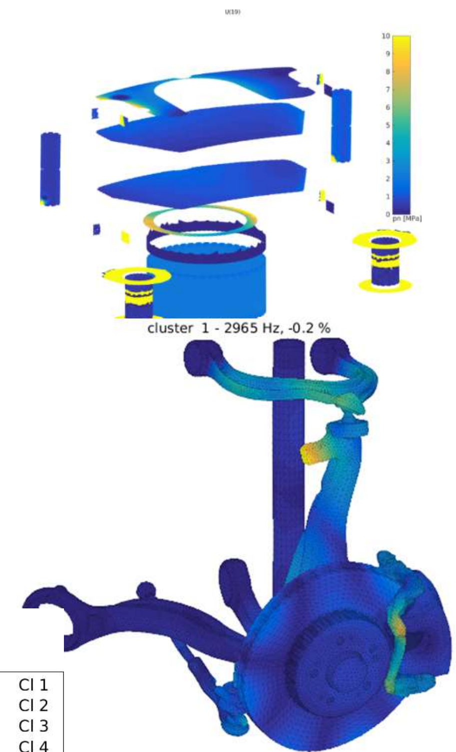




Reanalyis : computational example

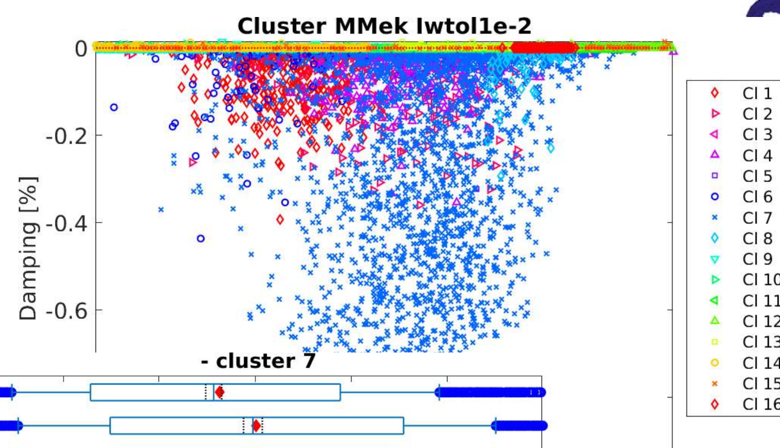
Squeal applications
- 8-15 components
- Multiple interfaces/parameters
- 300-600 modes

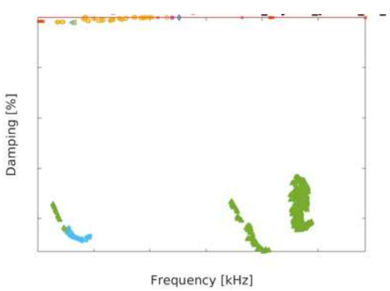


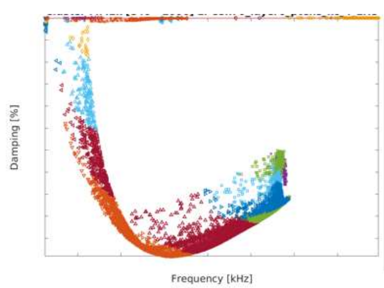


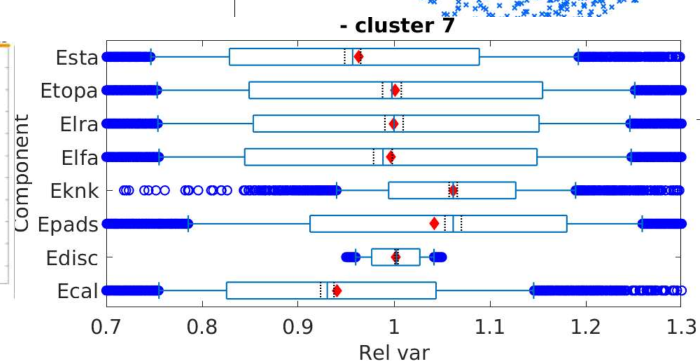
Design exploration 1000 points
- Full 80 days CPU, 22 TB
- CMT a few hours off-line learning, <1h exploration

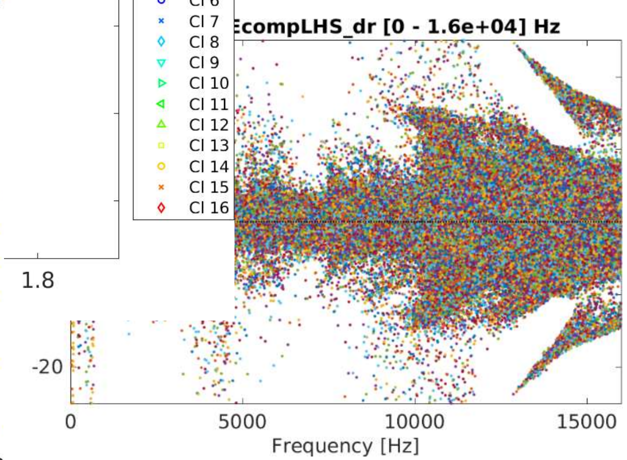
Exploitation
- DOE : basic edges, LHS
- Clustering & parameters











Families of reduced models

p	physical parameters (geometry, constit.)
$Z(p,s)=[M(p)s^2+K(p)]$	finite element model
$Z_R(p,s)=T^TZT$	reduced dynamic model
$T(p)=[\phi_1 \ldots \phi_{NR}\ {}_K K^{-1}b]$	modal model
$\{y\}=[H]\{u\}$	transfer functions

Kinematic reduction $\Leftrightarrow$ choice fixed Ritz basis $\{q\}=[T]\{q_R\}$

First order enhancement to MSE

$[Z(E(s),s)]\,\{q\} = \{F\}$ Damped viscoelastic resp. rewritten as

$$[Z(E_o,s)]\,\{q\} = \{F\} - \left[\sum (E(s)-E_o)/E_o[\mathrm{Im}(Z-Z_o)]\right]\{q\}$$

Tangent system, internal loads

Basis contains

- Modes to represent nominal resonances
- Flexibility to viscoelastic loads associated with nominal modes

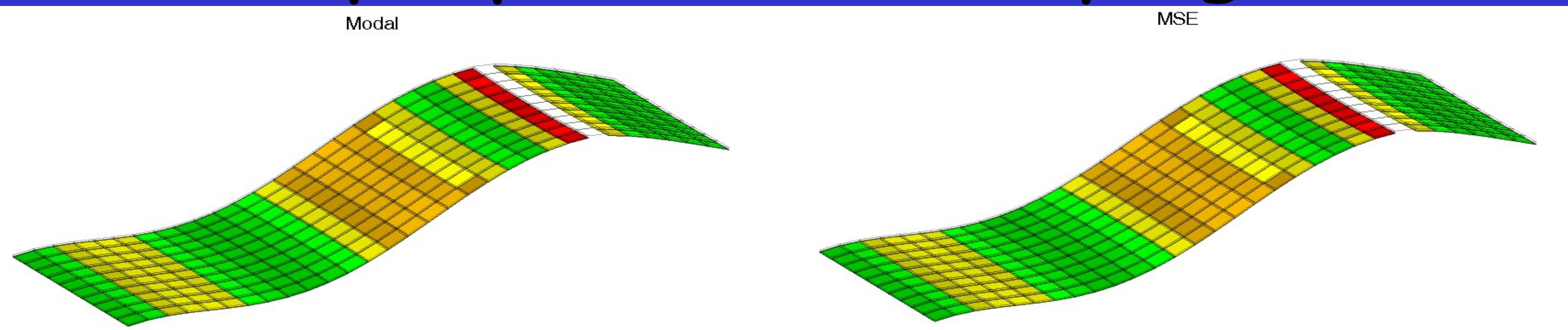
$$T = \left[\phi_{1:NM} \quad K_o^{-1}[\mathrm{Im}(Z-Z_o)]\,\phi_{1:NM}\right]$$

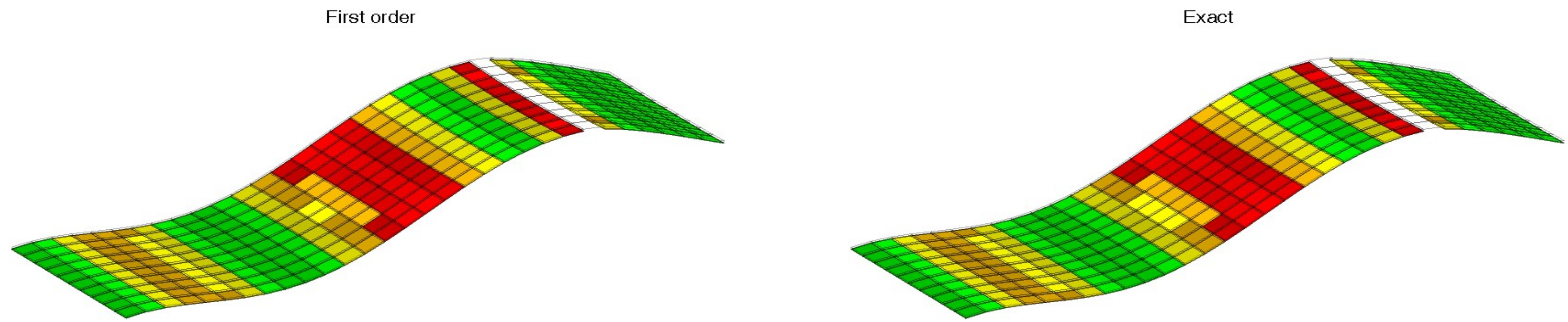
Modes MSE

static response to unit load

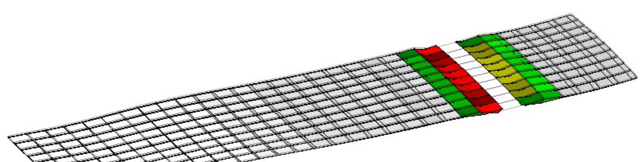
Principle of reduction (assumptions on excitation space & freq) unchanged

Non proportional damping model





- Forced response around a resonance



First order shape : $T = [K_o^{-1}\ [Im(Z-Zo)]\ \phi_4\ ]_{orth}$

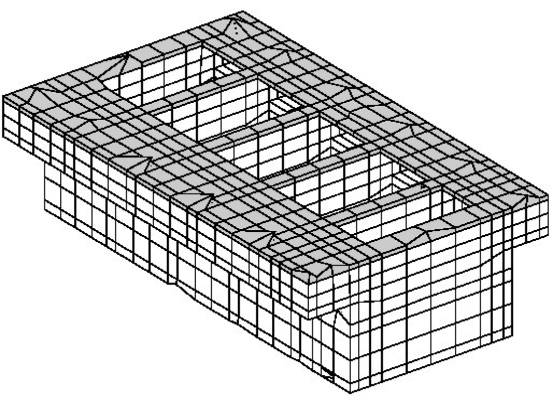
Reanalysis

Reanalysis

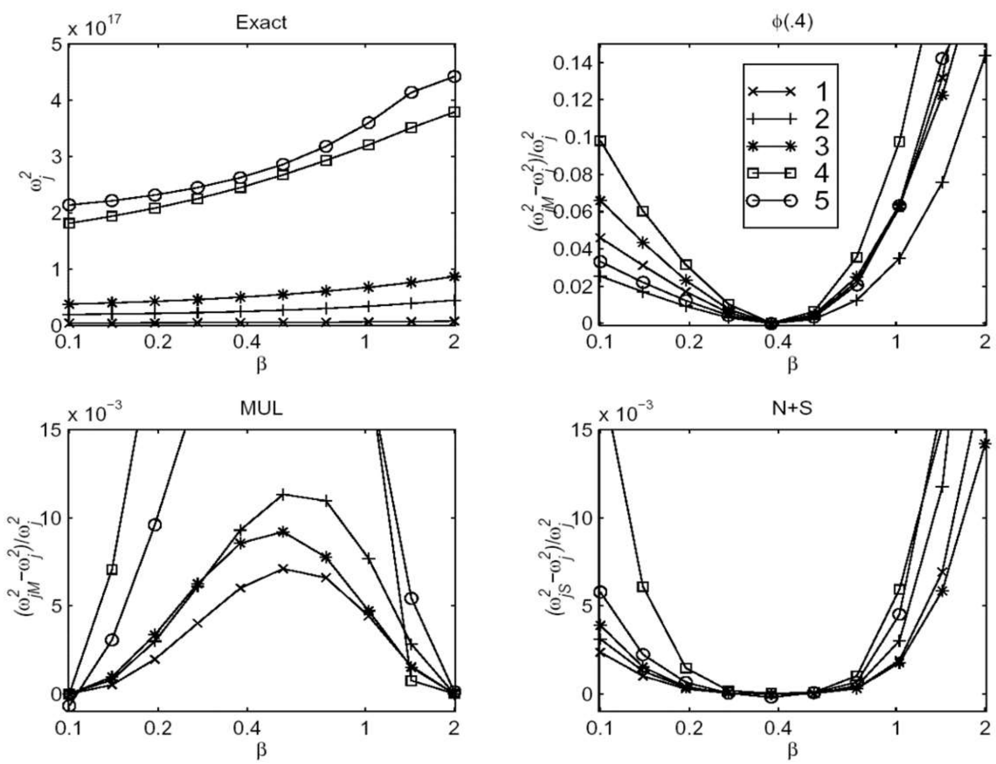
Multi-model bases

$$\left[\left[T^T K(p) T\right] - \omega_{jR}^2(p)\left[T^T M(p) T\right]\right]\{\phi_{jR}(p)\} = \{0\}$$

- Modes $T=[\Phi(p_0)]$
- Multi-model $T=[\Phi(p_1)\ \Phi(p_2)]$
- Modes + sensitivities $T=[\Phi(p_0)\ \partial\Phi(p_0)/\partial p]$

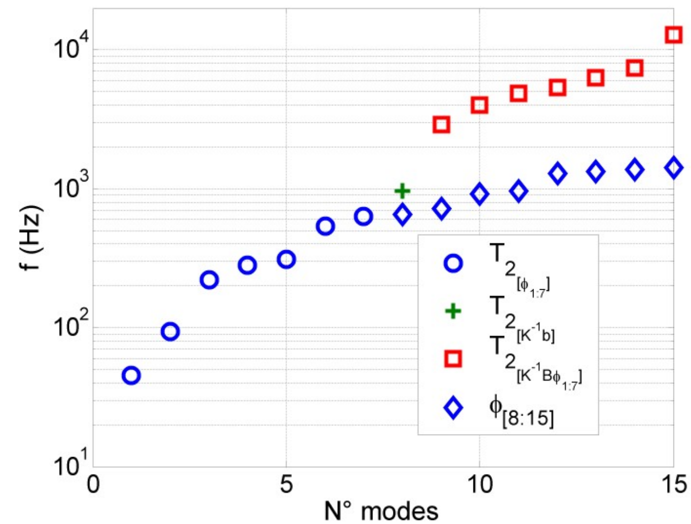


Frequencies



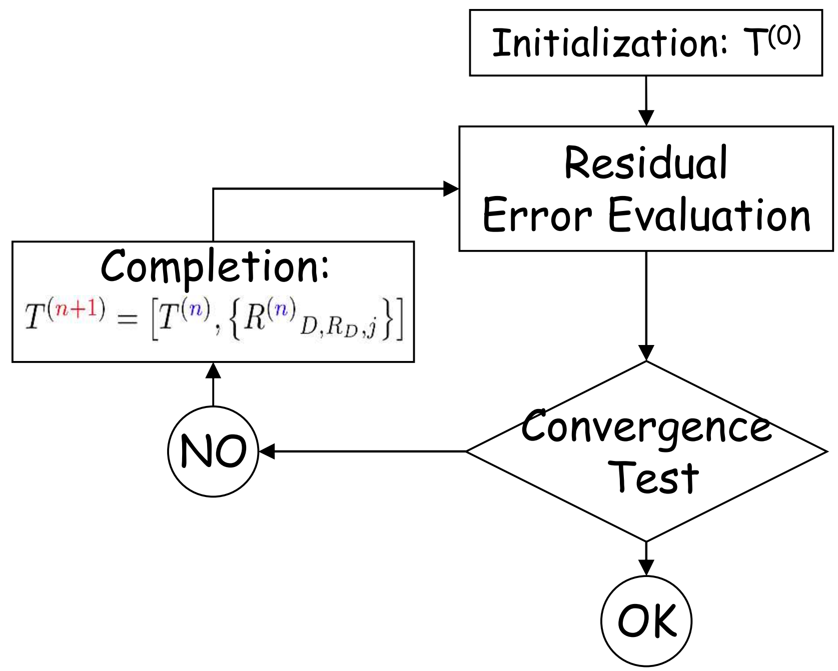
Reduction Bases/Starting Point

- Modal $T=[\Phi(p_0)]$
- Modal + static Responses to input $T=[\Phi(p_0)\ K^{-1}b]$
- Modal + static responses
  to representative loads
  $T=[\Phi(p_0)\ K^{-1}[b\ R_j]\ \Phi(p_k)]$



- Frequencies of corrections □ differ from modes ♦ : the added information is VERY different

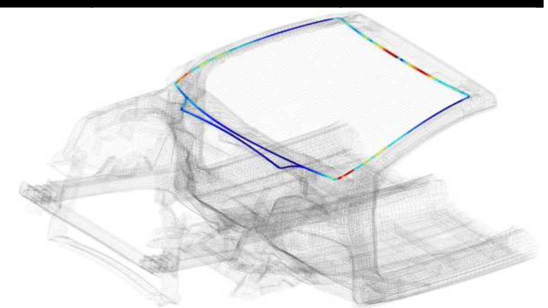
- Extension : residue iteration

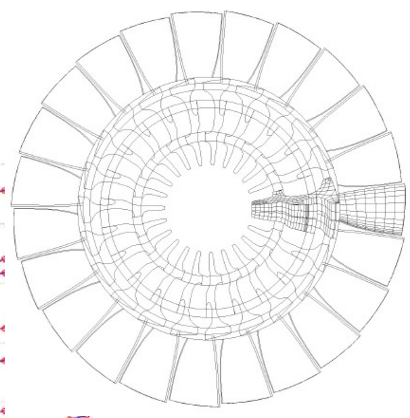


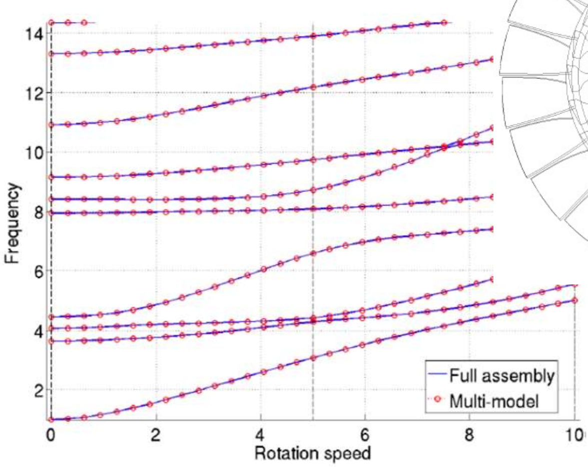
Fixed basis : enormous cost reduction

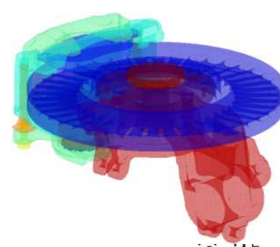
- Windshield joint complex modes at 500 design points for ½ cost of direct solver
- Campbell diagram : 200 rotations speeds for the cost of 4.
- Squeal instabilities as function of pressure : few pressures sufficient for interpolation

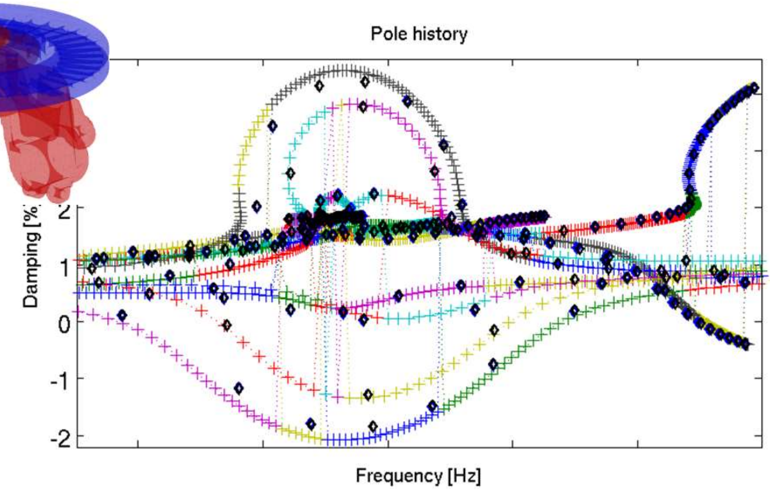
Ψ,λ	SOL107	2200s
Φ,ω	SOL103	300s
Ψ,λ Reduced	First order Error <4%	490s
Ψ,λ(500*T)	SOL107	~ 12 days
Ψ,λ(500*T) reduced	First order Error small	~1000s









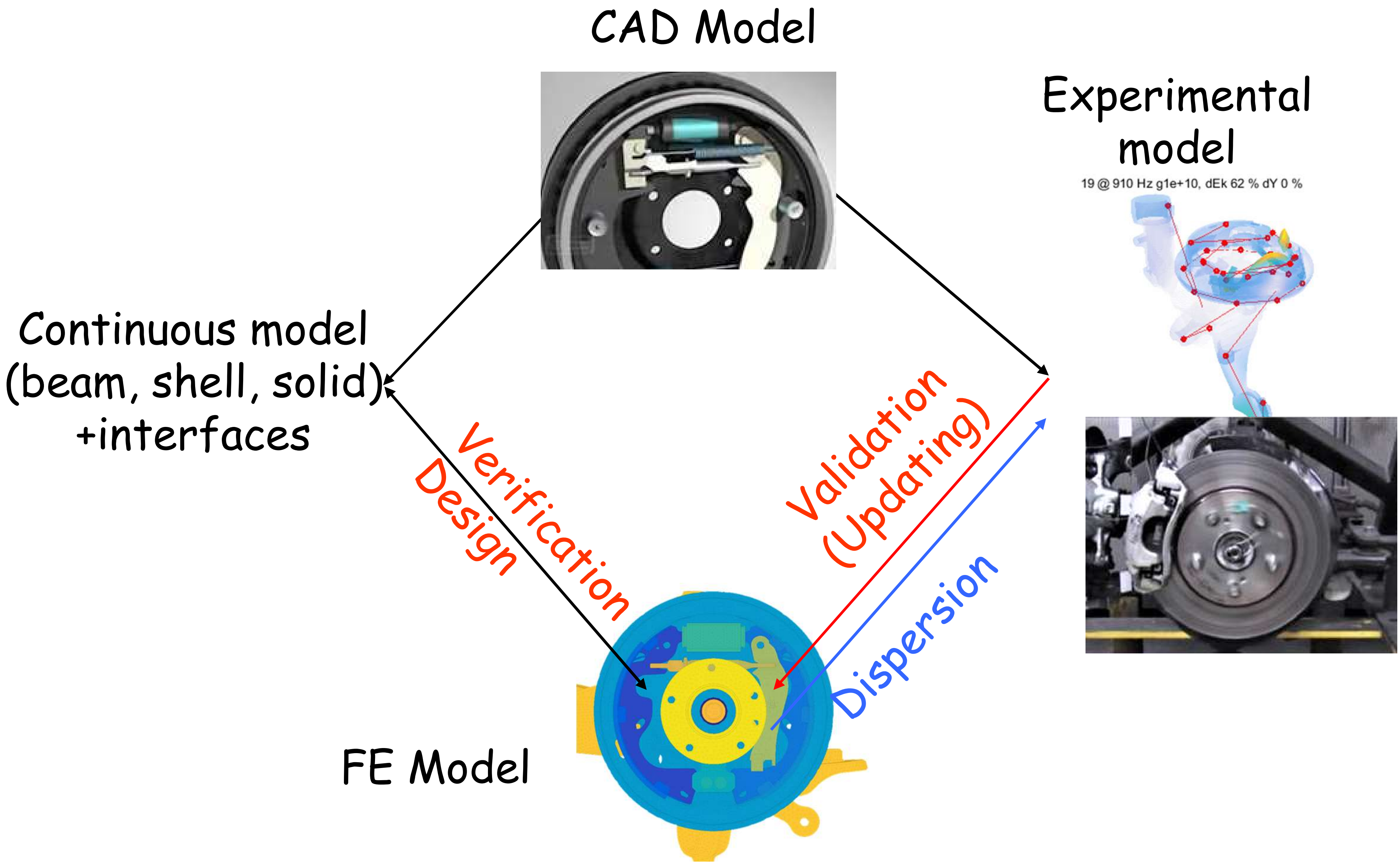


Updating

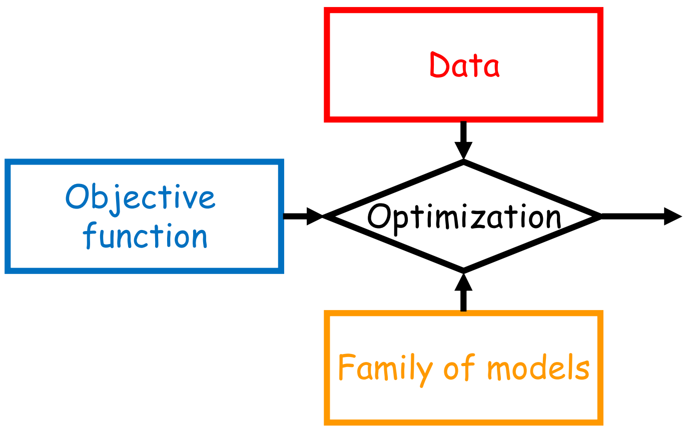
- Principles
- Matrix or parameter updating
- Objective function summary
- Parameter selection
- Optimization tools

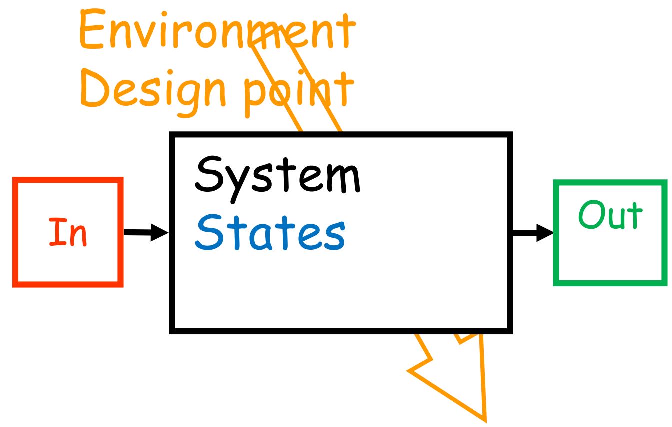
MS2SC http://savoir.ensam.eu/moodle/course/view.php?id=1874
PROVIR http://savoir.ensam.eu/moodle/course/view.php?id=490

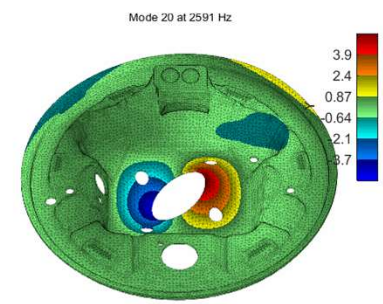
Model validation and verification



Parametric inverse problems

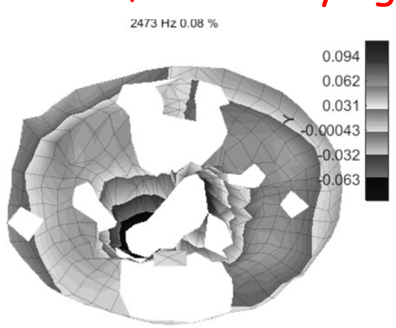




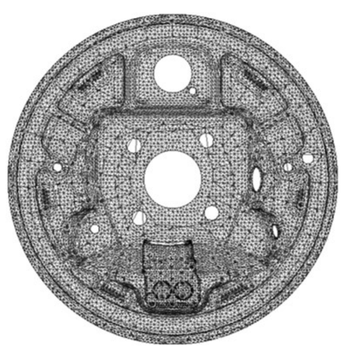


FEM error
- Geometry
- Material parameters
- Multi-scale problems & equivalent parameters
- Junction representation
- Design change & modal property tracking

Identification error
- Noisy measurements
- Identification bias
- NL, time varying, …

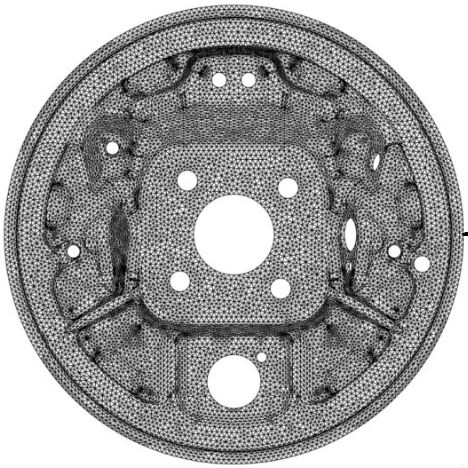


Topology errors
- sensor/act position
- matching

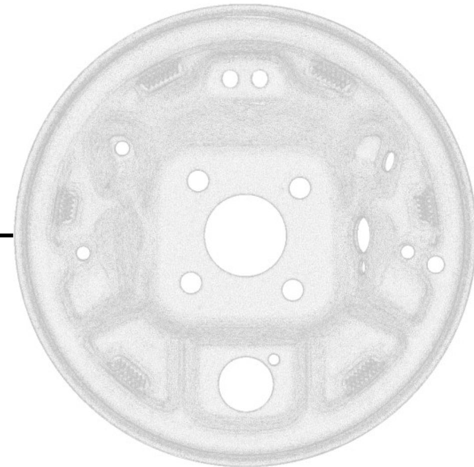


Geometry updating

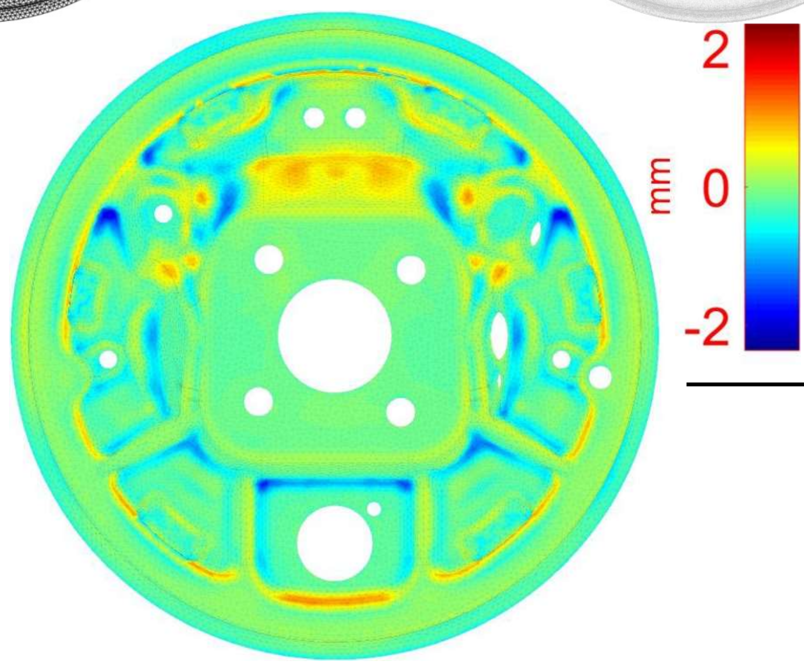
Nominal geometry



Measured geometry

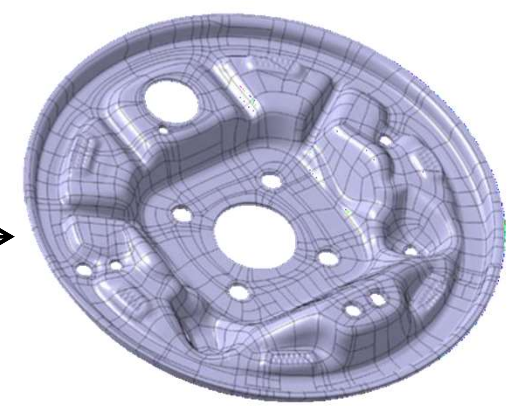


- Up to 2 mm on surface
- 11% error on volume



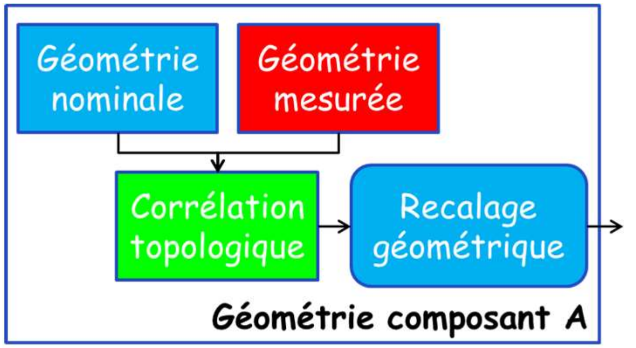
Geometry correlation

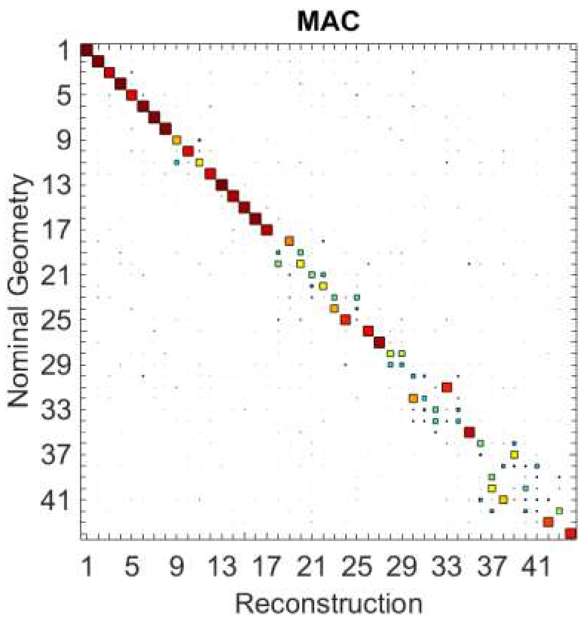
Inverse CAD
(Geomagic, …)
Morphing (SDT)

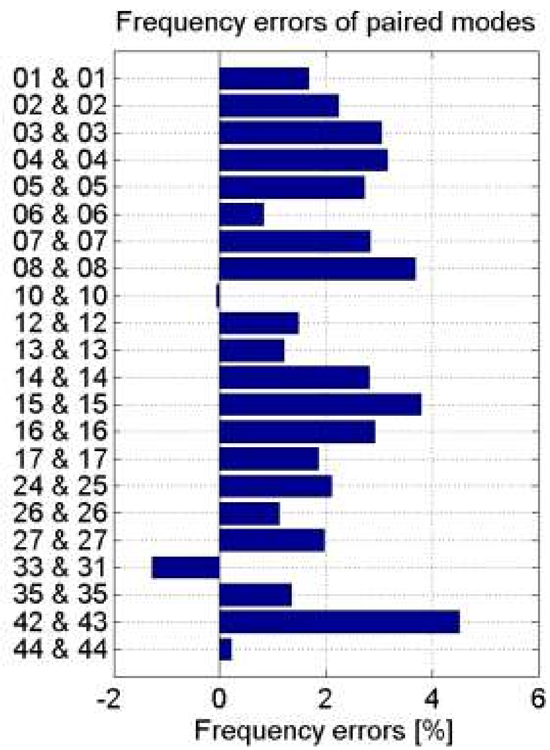


Geometry updating

- Frequency band : 0 – 6kHz
- Notable shape difference >3 kHz
- Frequency error
  - 2 % mean
  - 5 % max

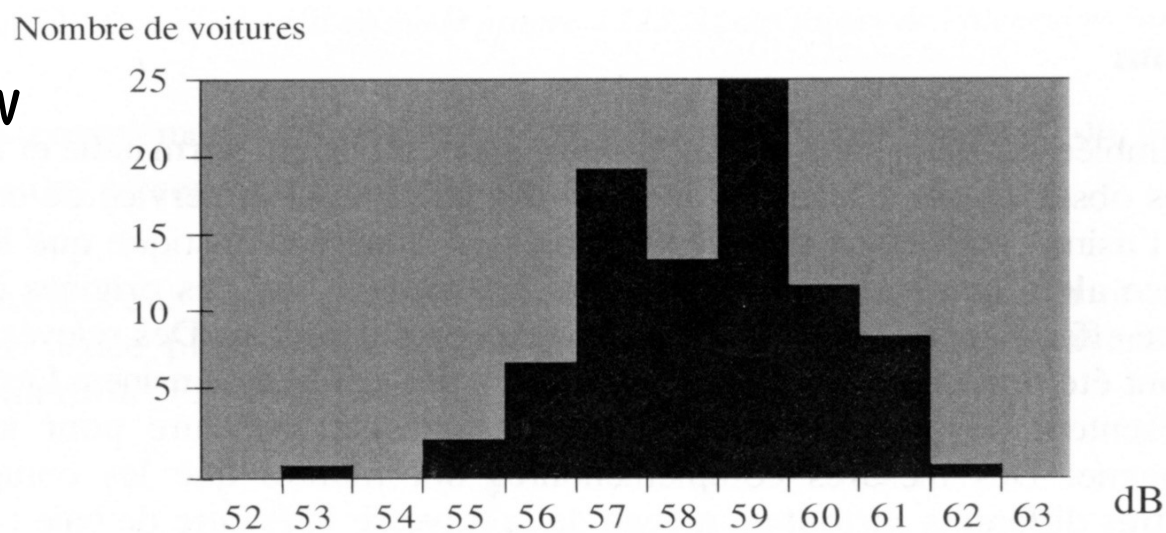






- Checking geometry should always be the first step
- Shells strong effect of geometry errors ≈ thickness

Invariance : influence of object



- Multiple measurements show significant variability

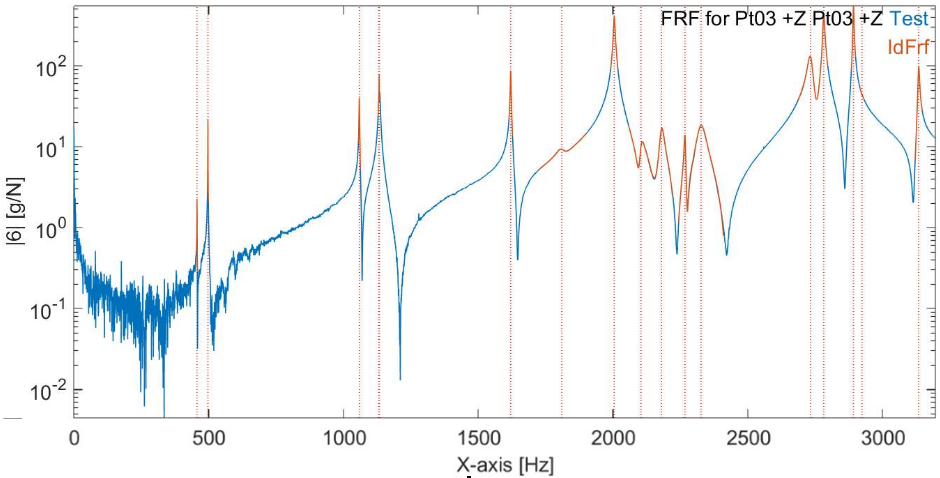
Spot welding manufacturing process has inherent variability due to

- Geometric tolerances
- Weld failure (several %)

Updating of material properties

Modal frequencies

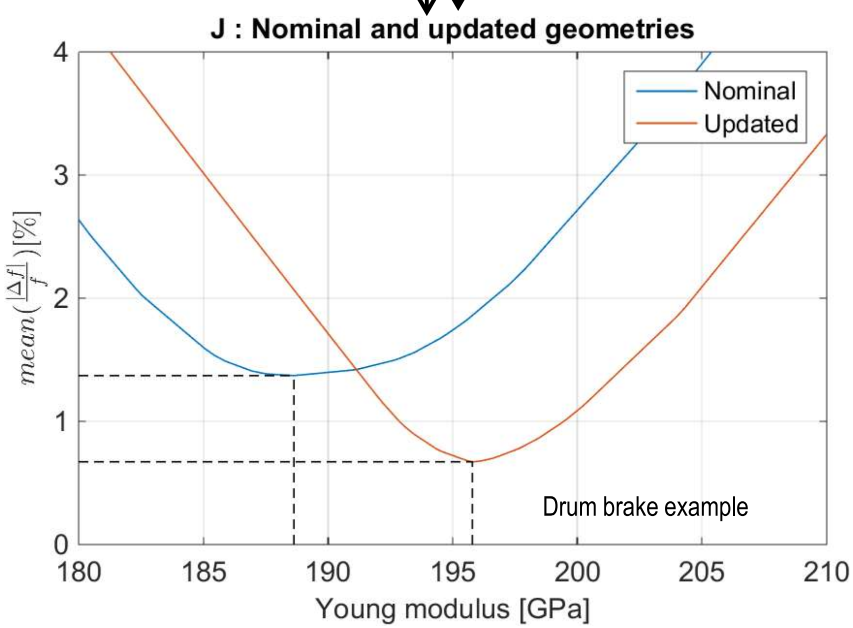
Weight



- Weight / volume ⇒ density
- Frequencies ⇒ modulus
- Test/FEM distance below physical dispersion (df/f≈2%)

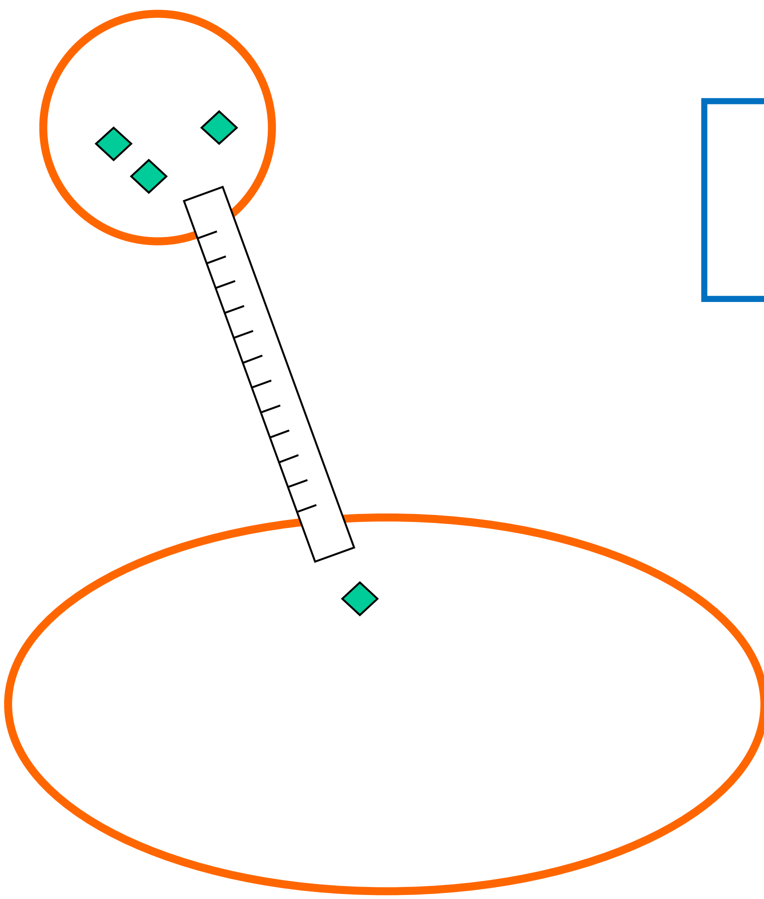
Frequency correlation

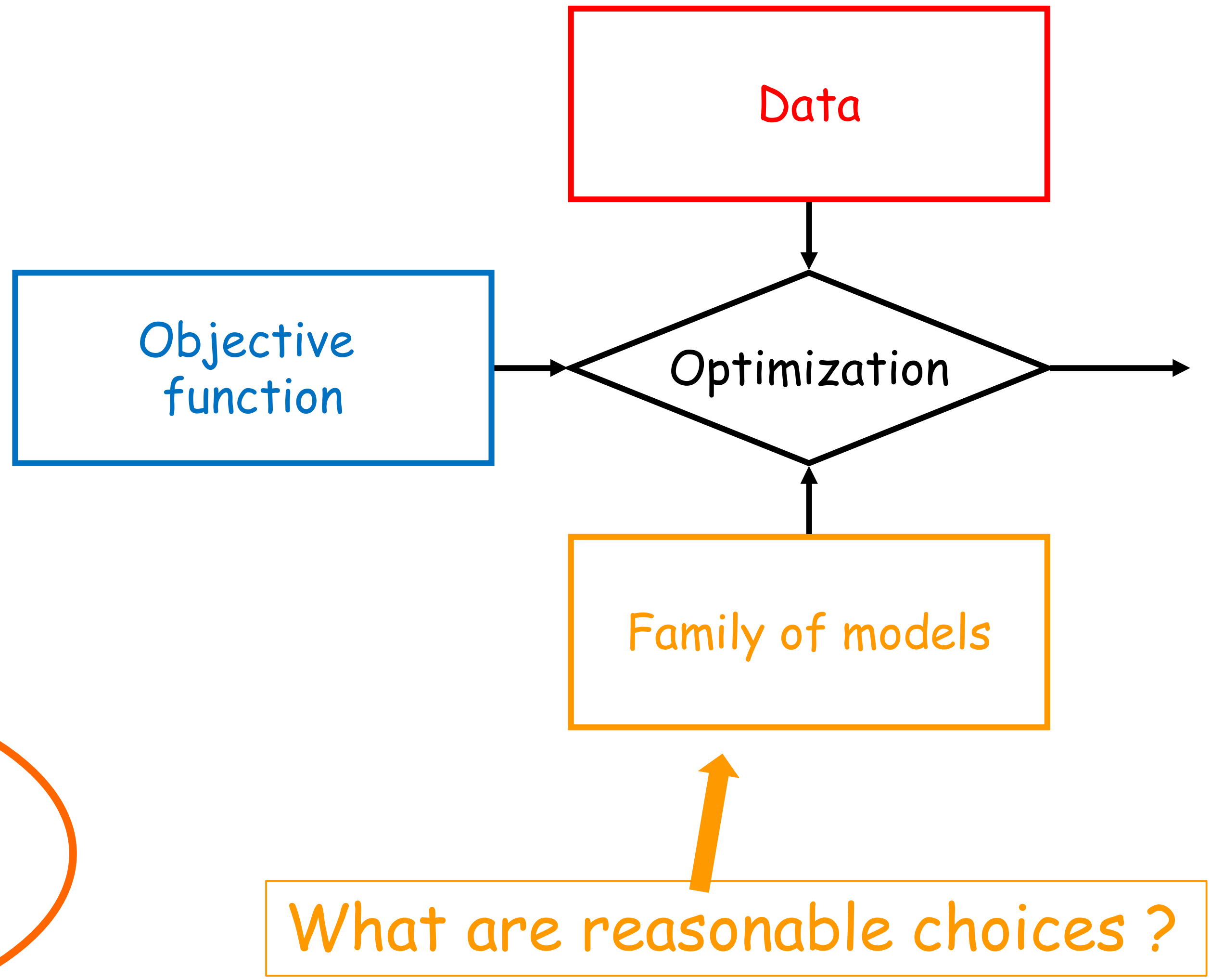
$$J = \min_E \left(\sum_i \frac{|\Delta f_i|}{f_i} \right)$$



Wrong volume = bias

Inverse problem : family of models





Equivalent materials / homogeneization

Family

- Micro : cell walls, glue, face-sheet, viscoelastic material
- Macro : shell/ orthotropic volume/ shell

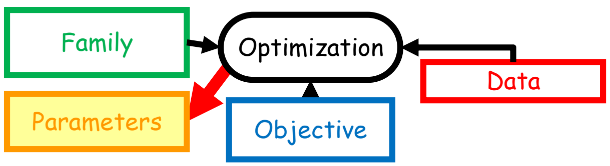
Objective

- Equivalence: waves/modes

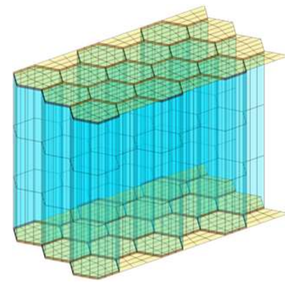
Parameters

- orthotropic law

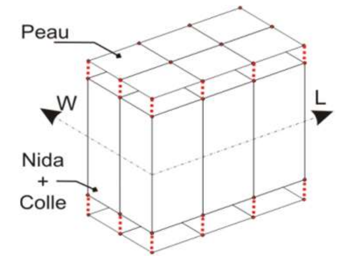
- Scale separation

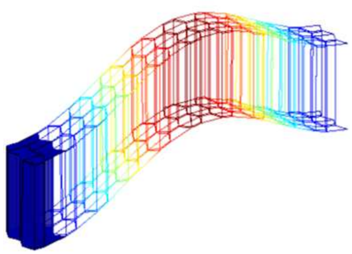


Detailed 3D honeycomb

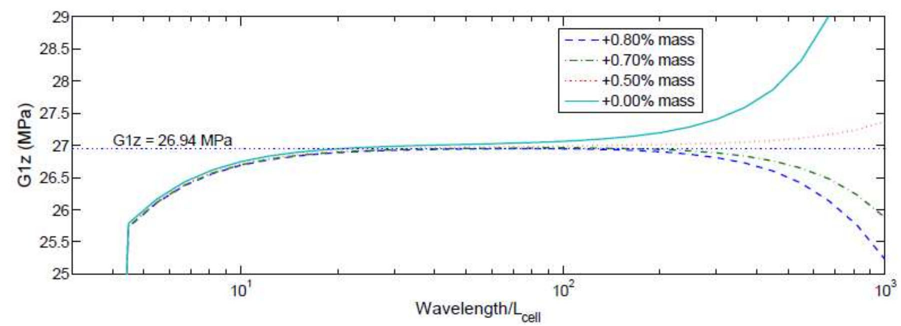


Shell/volume/shell

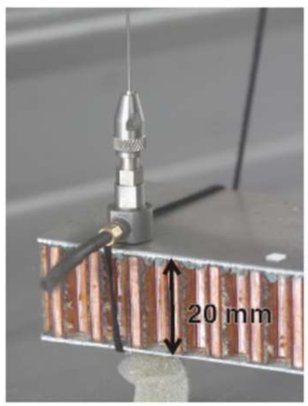


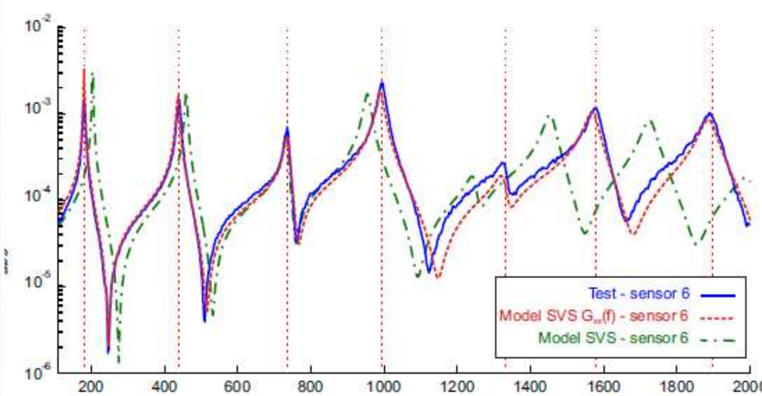


Numerical homogeneization



Updating from test



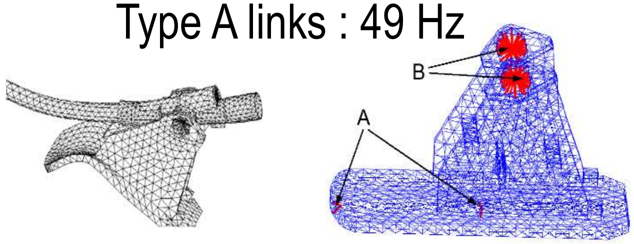


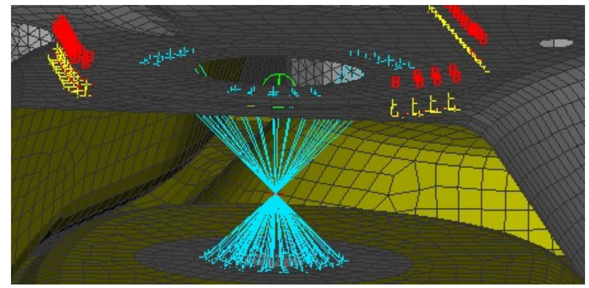
C. Florens, « Modeling of the viscoelastic honeycomb panel equipped with piezoelectric patches in view of vibroacoustic active control design », ECP, 2009.

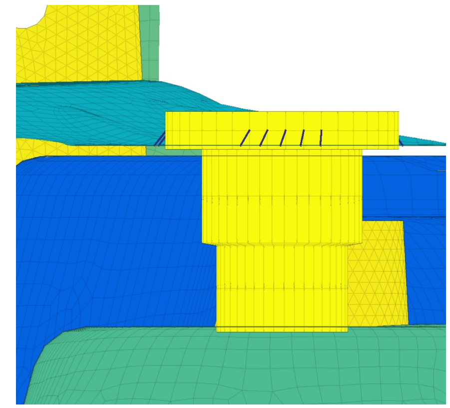
Junctions : contact / 0D strategies

Master point : mass & stiffness error

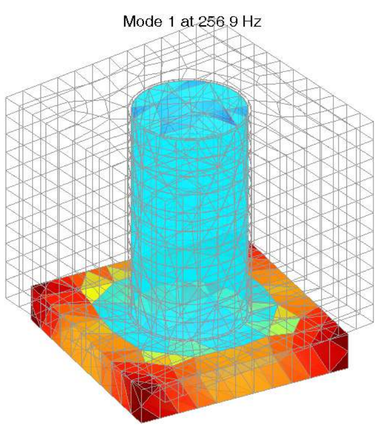
Type B links : 82 Hz
Type A links : 49 Hz

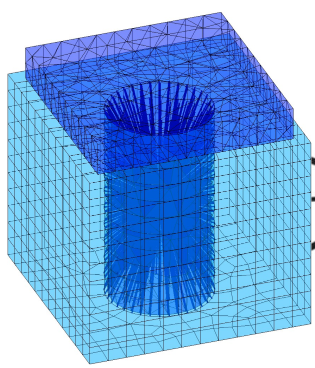


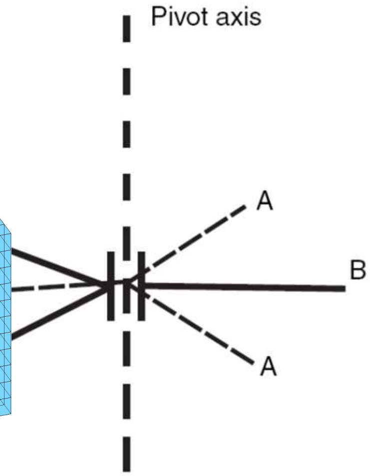




Kinematic junctions (pivots, …) often difficult with 2D

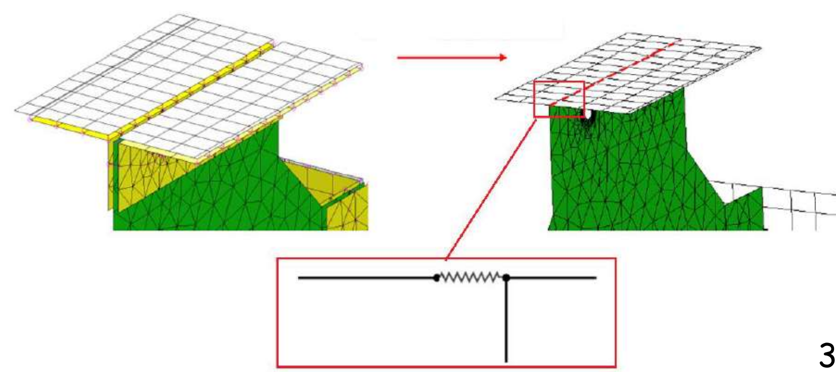






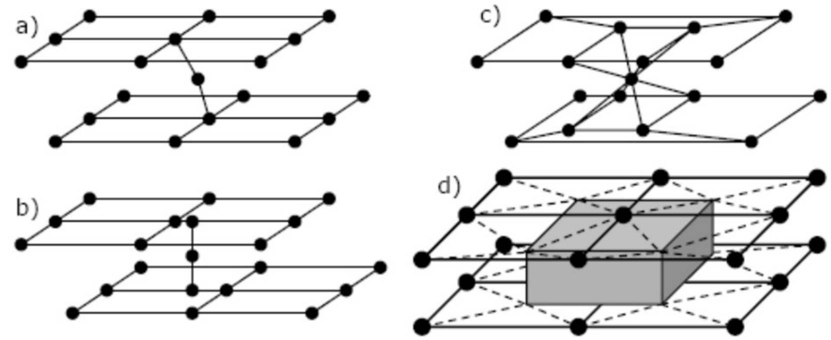
Contact : pivot mode at non zero frequency

Detail simplification (equivalent spring) may require complex numeric/test identification

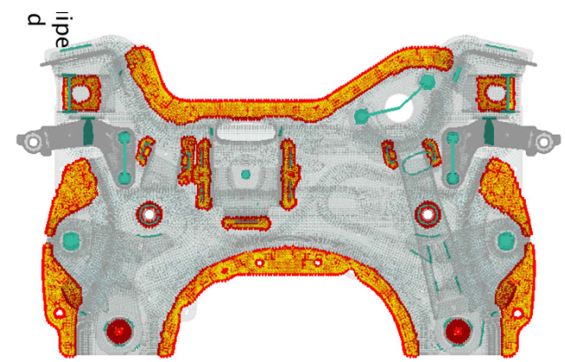


Junction problems : weld spots

- Weld spots [1]



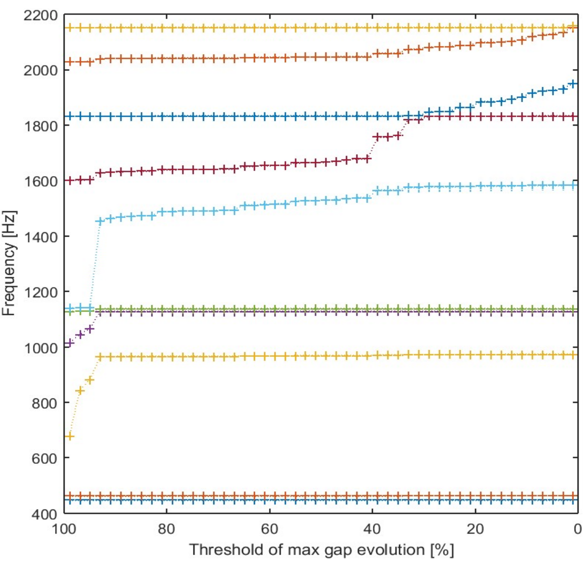
- Contact around WS [2]

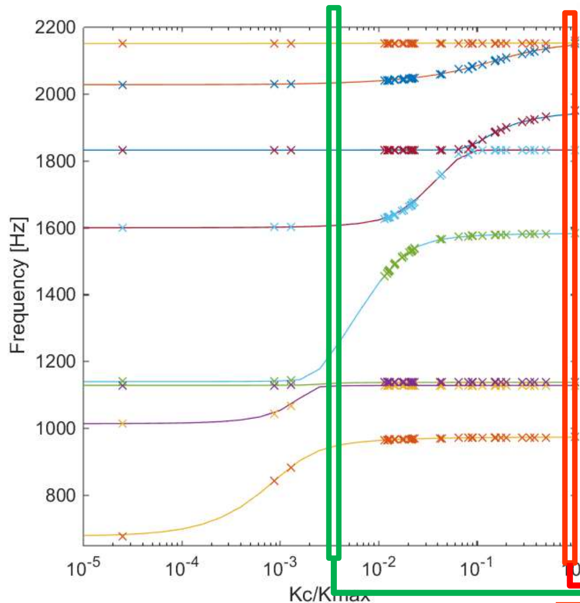


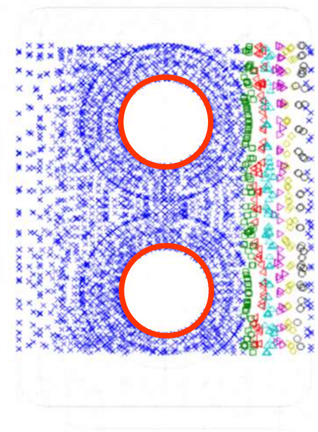
[1] Lardeur & All .: Spot weld modelling techniques and performances of finite element models for the vibrational behavior of automotive structures, ISMA 2000.
[2] G. Vermot Des Roches, E. Balmes, et S. Nacivet, « Error localization and updating of junction properties for an engine cradle model », in ISMA, 2016

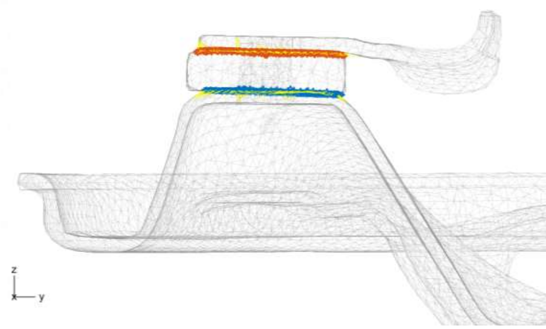
Junction : equivalent contact surface/stiffness

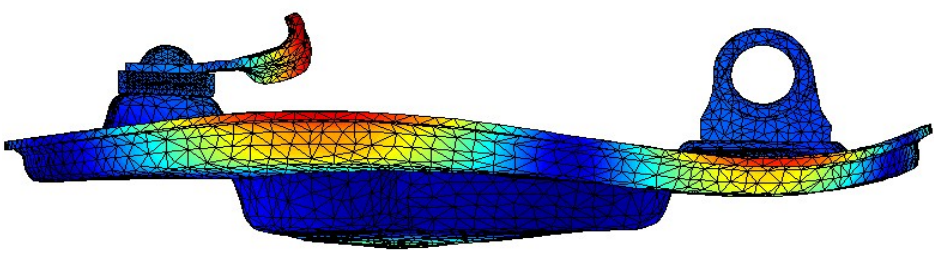
- Variable contact surface and stiffness
- Comparable impact frequencies and shapes

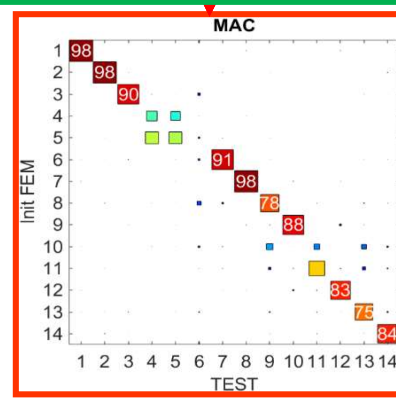


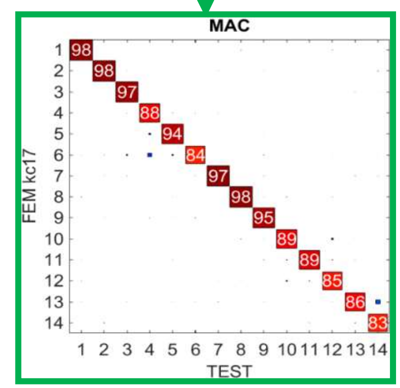






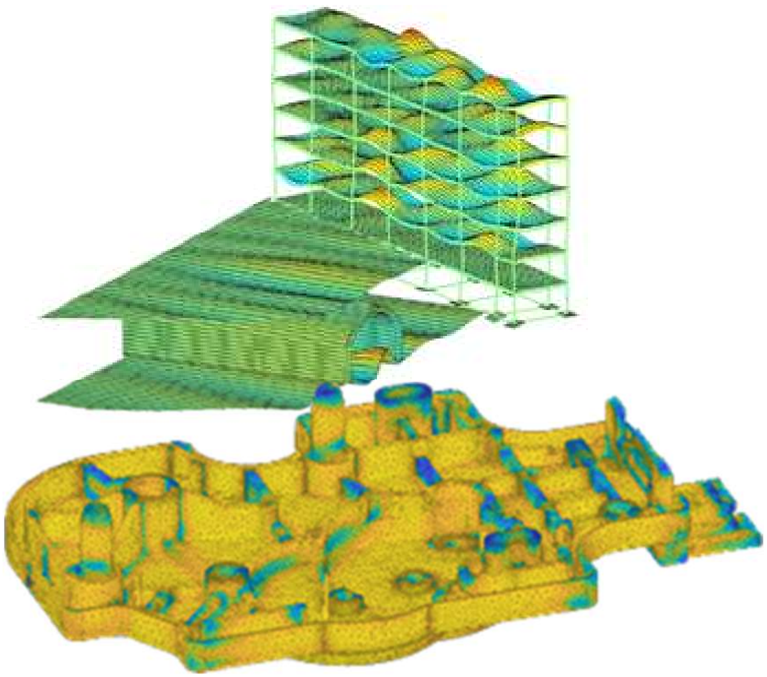


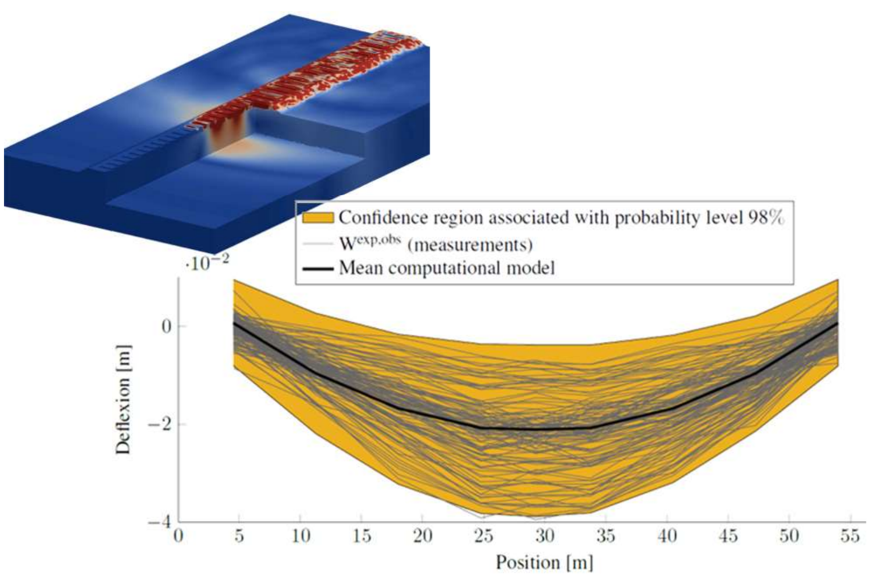




[1] G. Martin, E. Balmes, T. Chancelier, « Review of model updating processes used for brake components », in Eurobrake, 2015
[2] Y. Goth, H. Reynaud, « A bolt assembly parametric model », in ISMA, Leuven, 2016.

Unknown/random/variable characteristics





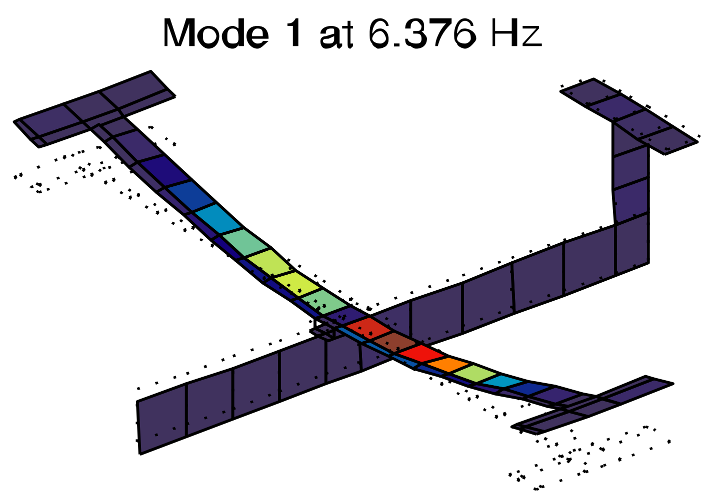
- Properties : soils, concrete, composite microstructure, ballast, …
- Loads (wind, waves, boiling water, …)

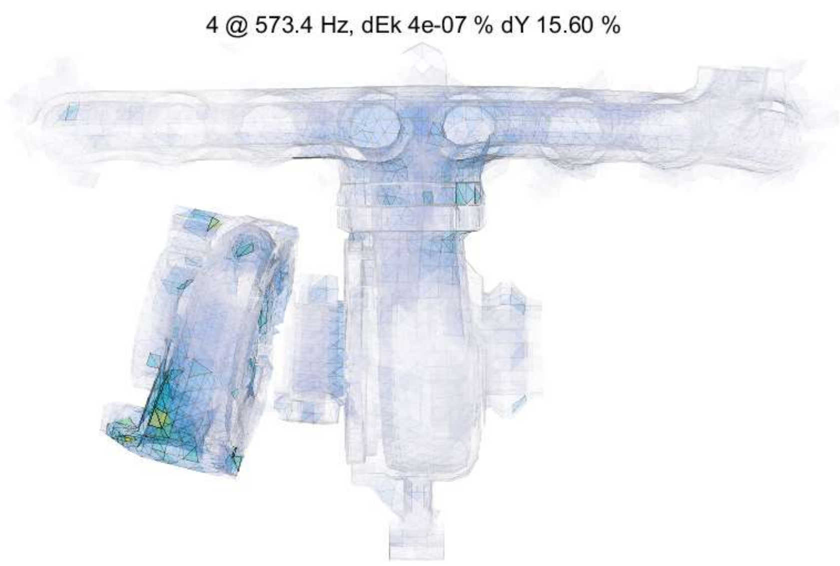
Needs

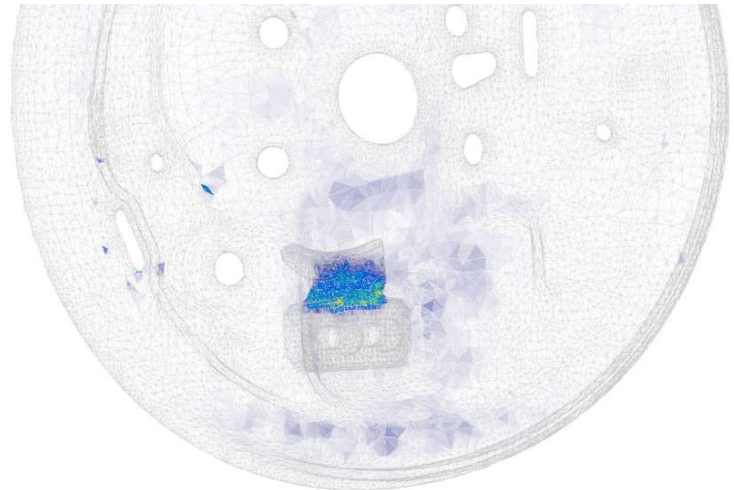
- Better knowledge (reduce uncertainties)
- A statistical description of uncertain fields
  - parametric (polynomial chaos, …)
  - Modal (non-parametric, see Soize & al.)
- A propagation method : from random model to random response

Measurement error -> localization

In MDRE expansion the energy distribution of the displacement residual ($R_d{}^{\wedge T} K^{(e)} R_d$) gives a localization







Updating & modeling errors

Usual modeling errors (solution : detailed model/equivalent par.)

- Incorrect model (joints, geometry, …)
- Variability of physical parameters : geometry, properties
- Un-modeled physics (damping, contact, non-linearities material and geometric, …)
- Discretization / mesh convergence / FEM parameter
- User input errors

Legitimate updating parameters

- Material parameters
- Equivalent macro-parameters : junction/contact stiffness
- Geometry

Next :

- Objectives
- Numerical tools for selection of parameters

Numerical methods for selection

- Objective is often of the form $J = || R ||^2$
  example frequency sensitivity

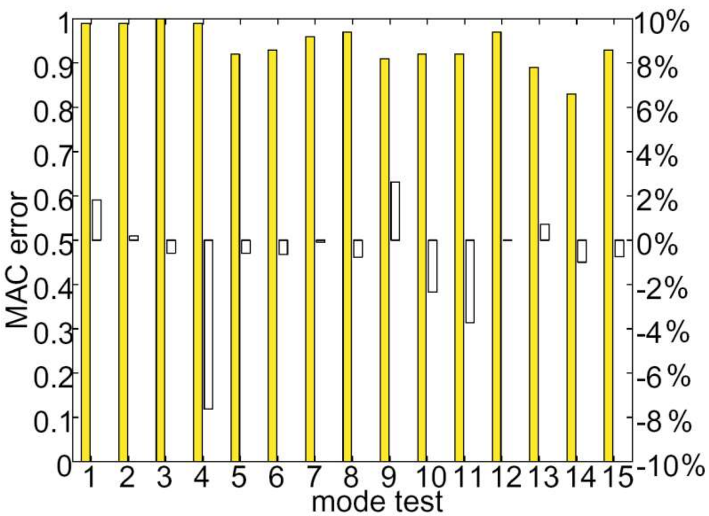
- Main effects
  - Parameter scaling
  - Parameter equivalence (poor conditionning)
  - Saturation (range where parameter is insensitive)
  - Parameter grouping (element divided in 2 $\Rightarrow$ sensitivity too)

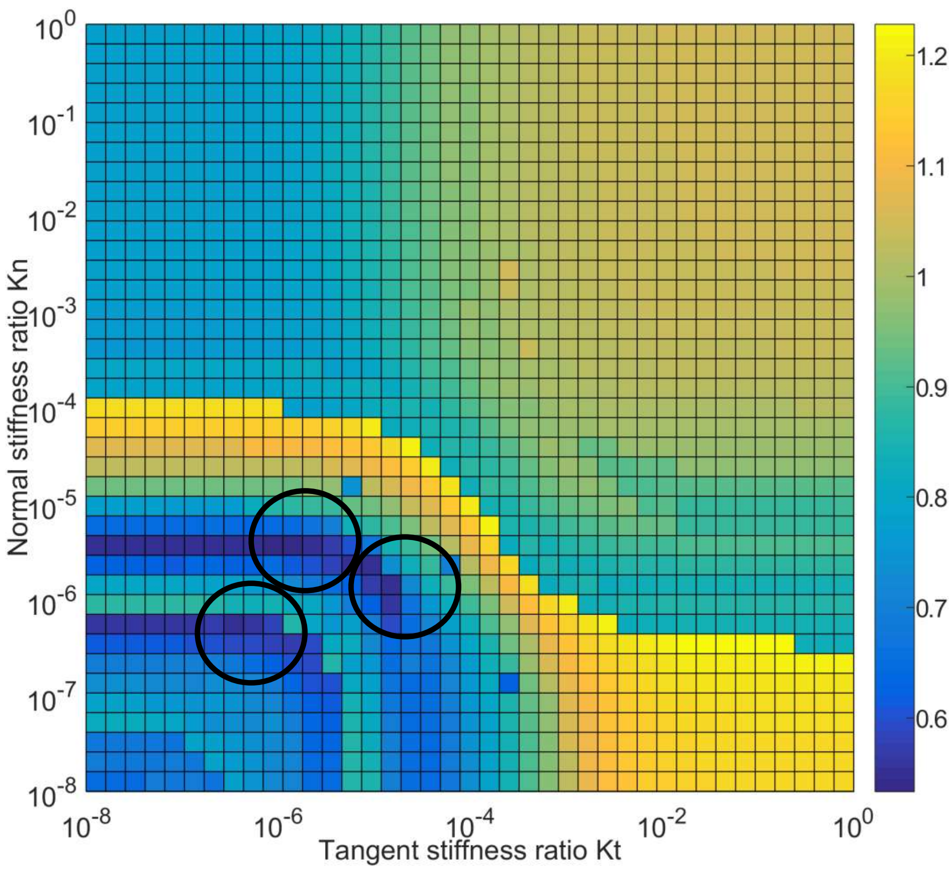
Freq-MAC objective

$$R_j(p) = \left\{ \begin{array}{c} \frac{f_{EF,i} - f_{Test,j}}{f_{Test,j}} \\ \beta(1 - MAC_{i,j}) \end{array} \right\}$$

$$J_{freq-MAC}(p) = \sqrt{\sum_{j=1}^{NM} \|R_j(p)\|^2}$$

Problems
- $J_{freq\text{-}MAC}$ irregular
- Jumps on mode pairing.
- Can be difficult to minimize





PhD. G. Martin, Méthode de corrélation calcul/essai pour l'analyse du crissement

Updating based on frequency and modeshapes

$$J(p) = \sum_{j=1}^{NR} a_j \left(\Delta\omega_j^2\right)^2 + \sum_{j=1}^{NR} b_j \left(\{\Delta c\phi_j\}^T \{\Delta c\phi_j\}\right)$$

$$\begin{bmatrix} \frac{\partial \omega_j^2}{\partial p_1}(p^n) & \cdots & \frac{\partial \omega_j^2}{\partial p_{N_p}}(p^n) \\ \frac{\partial c\phi_j}{\partial p_1}(p^n) & \cdots & \frac{\partial c\phi_j}{\partial p_{N_p}}(p^n) \end{bmatrix} \left\{ \begin{array}{c} \delta p_1^{n+1} \\ \vdots \\ \delta p_{N_p}^{n+1} \end{array} \right\} = \left\{ \begin{array}{c} \omega_{\mathtt{id}_j}^2 - \omega_j^2(p^n) \\ c\phi_{\mathtt{id}_j} - c\phi_j(p^n) \end{array} \right\}$$

- Variant MAC rather than relative error
- Problems : pairing, sensitivity, numerical conditioning, systematic test errors

Non linear least squares

$$J(p) = \|R\|_2^2 = \mathrm{Trace}\left(R^T R\right) = \sum_{i,j} \bar{R}_{ij} R_{ij}$$

- First derivative $$\frac{\partial J(p)}{\partial p} = 2\mathrm{Trace}\left(\frac{\partial R^T}{\partial p} R\right)$$
- Second derivative $$\frac{\partial^2 J(p)}{\partial p^2} = 2\mathrm{Trace}\left(\frac{\partial R^T}{\partial p}\frac{\partial R}{\partial p} + R^T \frac{\partial^2 R}{\partial p^2}\right)$$
- Newton method $p^{n+1}=p^n+\delta p^{n+1}$ with
- Convergence $$\left[\frac{\partial^2 J(p)}{\partial p^2}\right]\left\{\delta p^{n+1}\right\} + \left\{\frac{\partial J(p)}{\partial p}\right\} = \{0\}$$

$$\lambda_{\max}\left(\left[\frac{\mathrm{d}^2 J}{\mathrm{d}p^2}(p)\right]\right) < 1$$

Sensitivity method

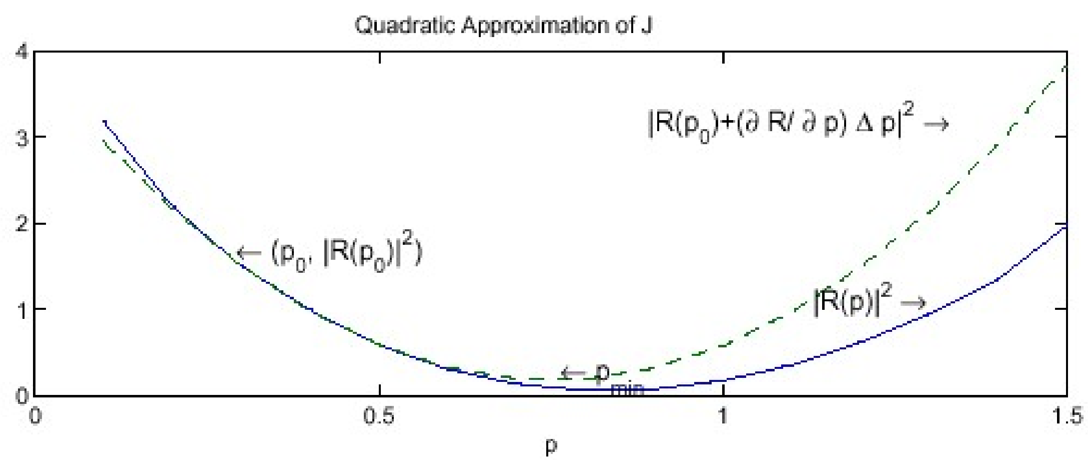
- Approximation of second order derivative $\Rightarrow$

$$\mathrm{Trace}\left(\left\{\frac{\partial R}{\partial p}\right\}^T \quad \left(\frac{\partial R}{\partial p}\left\{\delta p^{n+1}\right\} - R(p^n)\right)\right) = 0$$

- Equivalent to (with appropriate norm)

$$\min_{\{\delta p^{n+1}\}} \left\| \left\{\frac{\partial R}{\partial p}\right\}\left\{\delta p^{n+1}\right\} - \left\{R(p^n)\right\} \right\|$$

- Standard least-squares form

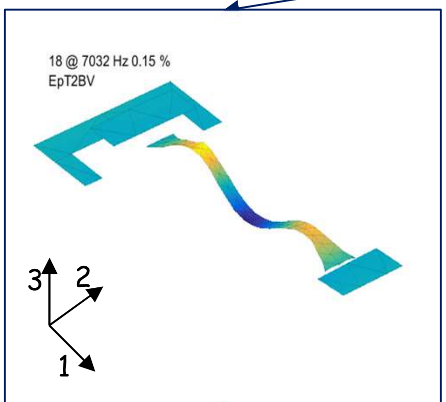


Parameter equivalence / conditioning

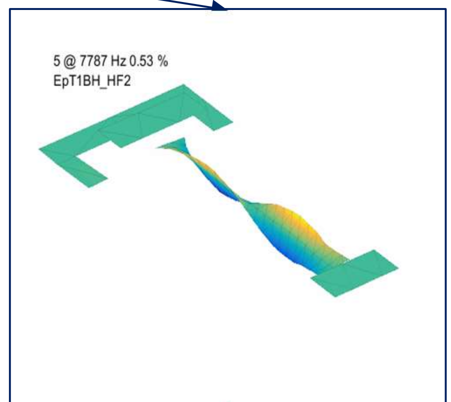
Tests

Observable modes

Vertical bending



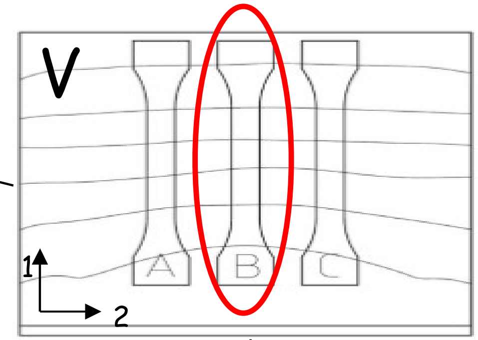
Torsions



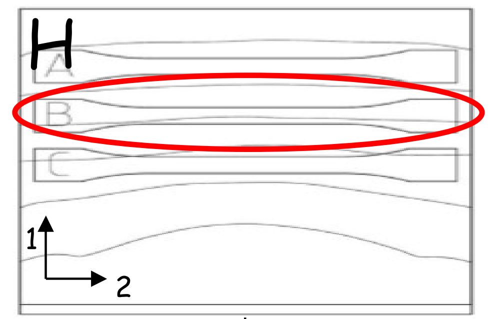
Identification

$$C^{-1} = S = \begin{pmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{pmatrix}$$

2 samples V/H



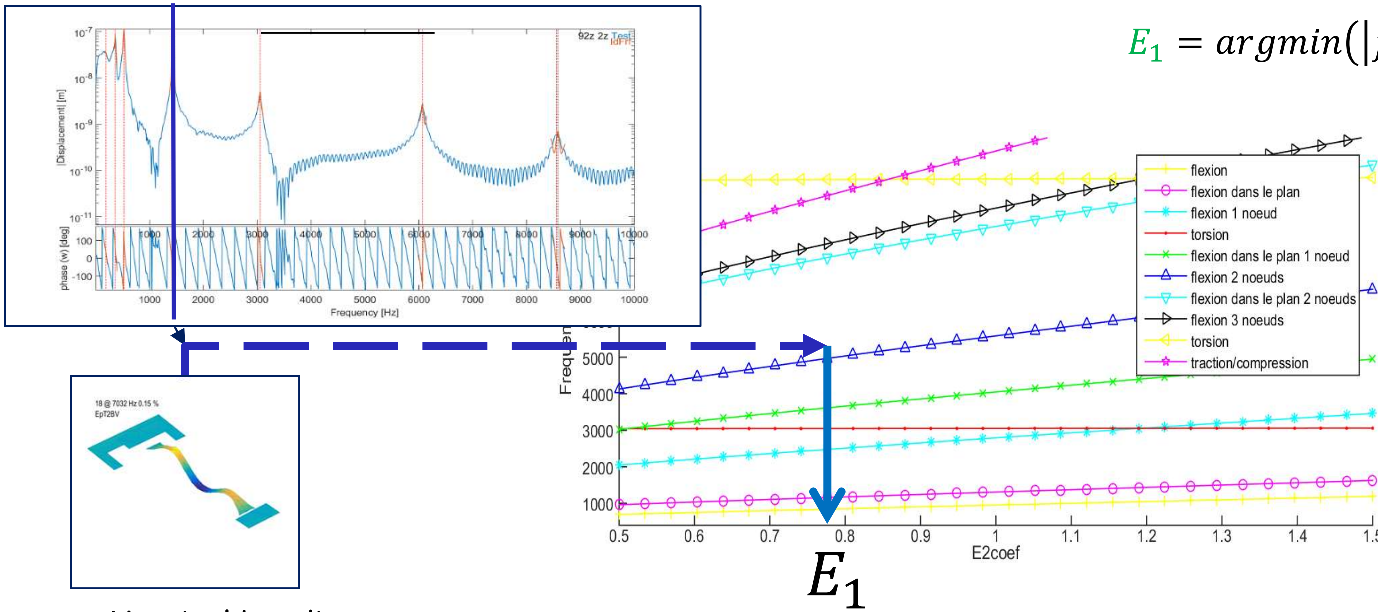
Identification of E_1 , G_{31} et G_{12}



Identification of E_2 et G_{23} $(+G_{12})$

Parameter equivalence : material example

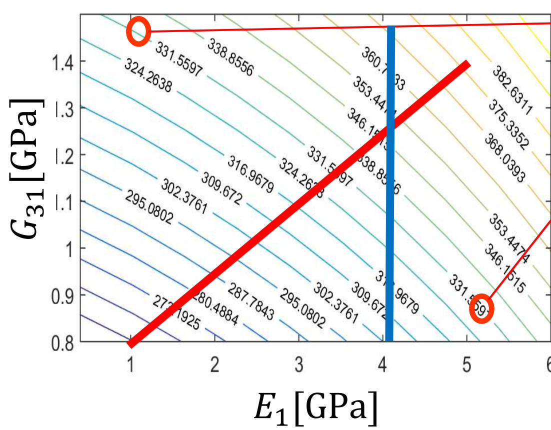
$E_1 = argmin(|f_{flexion}(E_1, G_{31}) - f_{flexion,obj}|)$



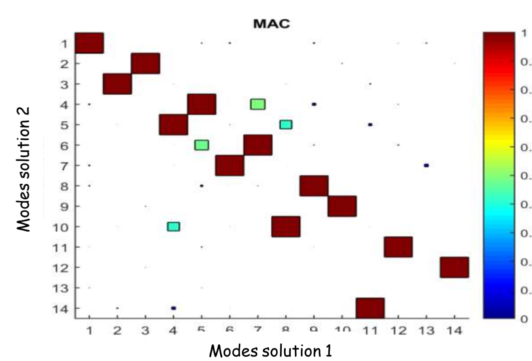
Vertical bending

Assumption must be made to improve conditioning

- G_{31}/E_1 constant ?
- E_1 traction ?



Solution is not unique
Mode-shape very similar



Least squares conditioning

- One solves LS

$$\min_{\{x\}} \|[A] \ \{x\} - [b]\|_2^2$$

- But errors

$$([A] + [\delta A]) \ (\{x\} + \{\delta x\}) \ = \ (\{b\} + \{\delta b\})$$

- Problem is well conditioned if :

$$\frac{\|\delta A\|}{\|A\|} \ll 1, \qquad \frac{\|\delta b\|}{\|b\|} \ll 1 \qquad \Longrightarrow \qquad \frac{\|\delta x\|}{\|x\|} \ll 1$$

- One can prove that (k condition number)

$$\frac{\|\delta x\|}{\|x\|} \ \leq \ \kappa(A) \left(\frac{|||\delta A|||}{|||A|||} + \frac{\|\delta b\|}{\|b\|} \right)$$

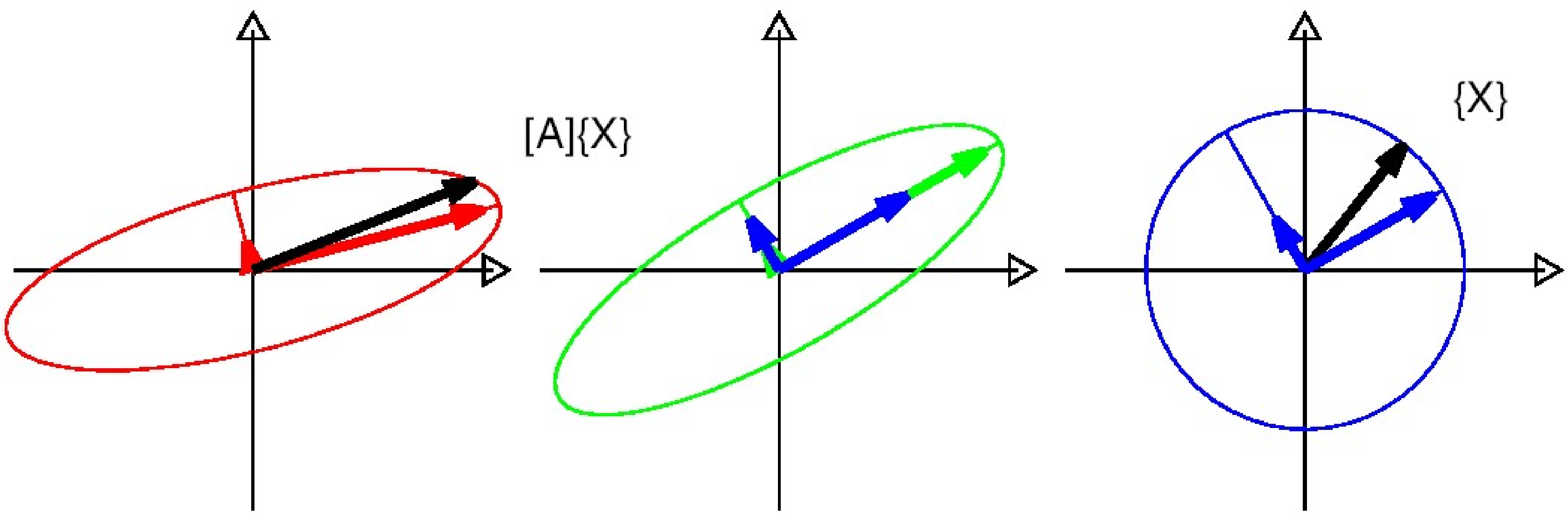
$$\kappa(A) \ = \ |||A||| \ |||A^{-1}|||$$

SVD (1/2)

A = U S V^H

Principal directions in the effect of operator A

$$\left[HH^H\right]\{U_j\} = \sigma_j^2\{U_j\} \qquad \left[H^HH\right]\{V_j\} = \sigma_j^2\{V_j\}$$



http://ocw.mit.edu/OcwWeb/Mathematics/18-06Spring-2005/CourseHome/index.htm

SVD (2/2)

Matrix form A = U S V^H

Series form

$$[A] = \sum_{j=1}^{\min(n,m)} \sigma_j \{U_j\} \{V_j\}^T$$

Standard properties

$$\sigma_{\max}(A^{-1}) = \frac{1}{\sigma_{\min}(A)}$$

$$\sigma_{\min}(A^{-1}) = \frac{1}{\sigma_{\max}(A)}$$

$$\sigma_{\max}(A) - 1 \leq \sigma_{\max}(I + A) \leq \sigma_{\max}(A) + 1$$

$$\sigma_{\min}(A) - 1 \leq \sigma_{\min}(I + A) \leq \sigma_{\min}(A) + 1$$

$$\sigma_{\max}(A + B) \leq \sigma_{\max}(A) + \sigma_{\max}(B)$$

$$\sigma_{\max}(AB) \leq \sigma_{\max}(A)\sigma_{\max}(B)$$

Least squares and SVD

- SVD of A is of the form

$$[A] = \sum_{j=1}^{\min(n,m)} \sigma_j \{U_j\} \{V_j\}^T$$

- Least squares solution given by

- This is

$$\{x\} = \sum_{j=1}^{\min(n,m)} \sigma_j^{-1} \{V_j\} \{U_j\}^T [b] = [V] \left[\diagdown \sigma_j^{-1} \diagdown \right] [U] [b]$$

$$\begin{aligned}
[A^+]\ [A]\ [A^+] &= [A^+] \\
[A]\ [A^+]\ [A] &= [A] \\
([A]\ [A^+])^T &= ([A]\ [A^+]) \\
([A^+]\ [A])^T &= ([A^+]\ [A])
\end{aligned}$$

SVD and conditioning

- For $||\delta A||=\varepsilon$ smaller eigenvalues need to be neglected (because not well known)

$$[\Sigma] \;=\; \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_{NR} & & & \\ & & & \epsilon & & \\ & & & & \ddots & \\ & & & & & \epsilon \end{bmatrix}$$

- Hence well conditioned pseudo-inverse

$$\{x\} \;=[A]^{+}\,[b] = \;\left[\sum_{j=1}^{NR} \sigma_j^{-1}\,\{V_j\}\,\{U_j\}^T\right][b]$$

Large scale and sensitivity saturation

- Structural parameters typically have saturation ranges for both low and high values
- Sensitive range needs to be determined first