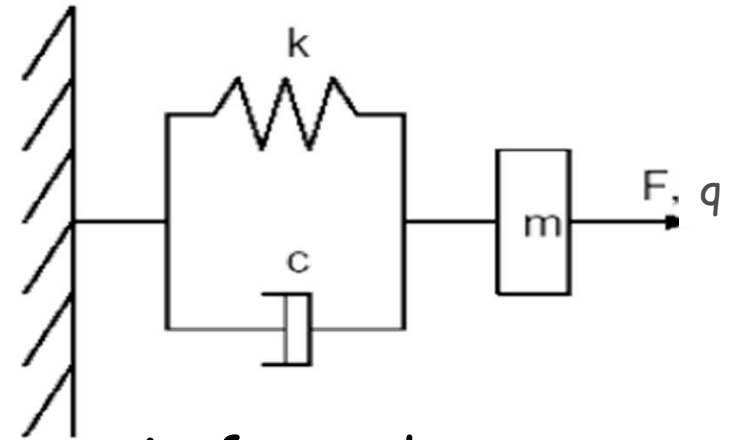


- 1 DOF ([chap.1, sec 2.1](#), course 1, also [Modal.pdf](#))
1. Transfer (time, Fourier $i\omega$ /Laplace s , asymptotic prop, NL)
 2. Poles, resonance, damping ratio -3 dB method
 3. States, state-space models, poles
 4. 1 DOF time exponential, convolution, logarithmic decrement
 5. Strategies for transient in time & frequency

Resonance (1 DOF oscillator), poly ch1

Dynamic equation :

$$m\ddot{q}(t) + c\dot{q}(t) + kq(t) = F(t)$$



Harmonic excitation

$$F(t) = \text{Re}(F(\omega)e^{i\omega t})$$

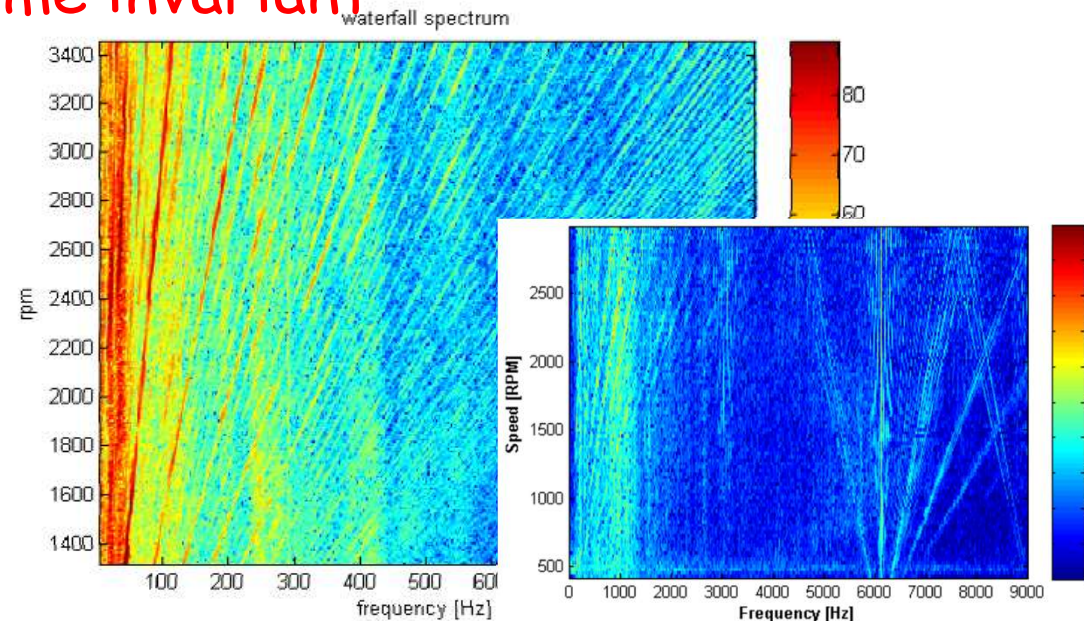
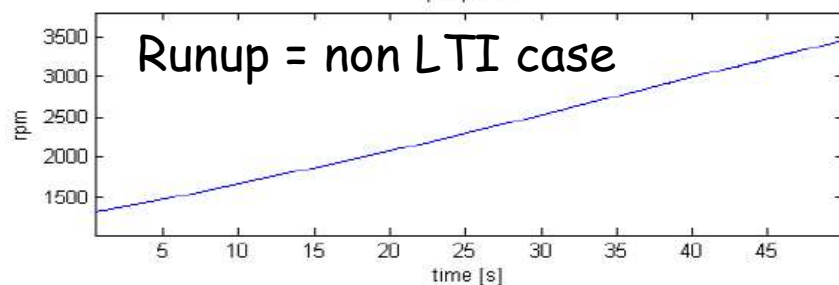
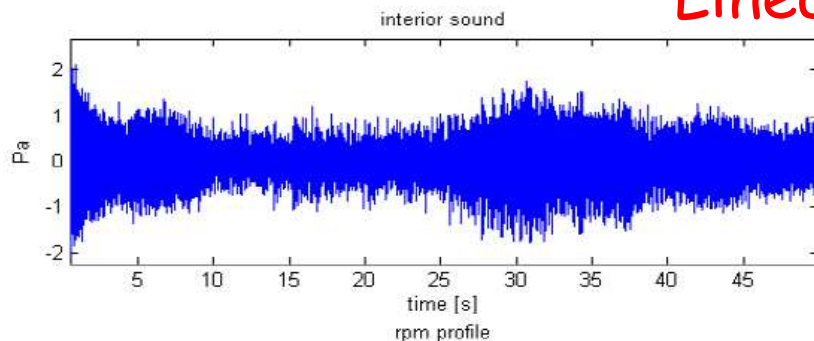
Hyp ?



Harmonic forced response

$$q(t) = \text{Re}(q(\omega)e^{i\omega t})$$

Linear time invariant



1 DOF time integration : state space (poly 3.1)

- Second order (meca : Abaqus, NASTRAN, ANSYS)

$$[M] \{\ddot{q}(t)\} + [C] \{\dot{q}(t)\} + [K] \{q\} = \{F(t)\}$$

- First order (ODE : Simulink, Simpack, ...)

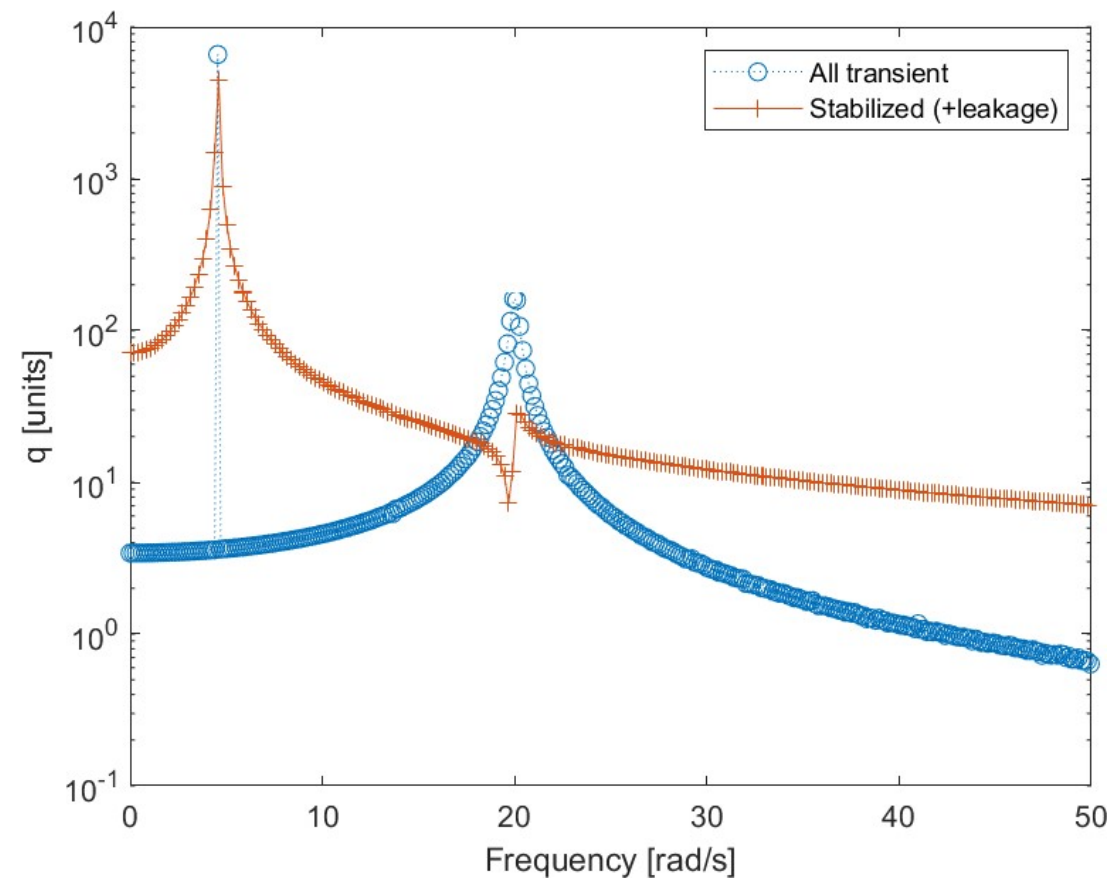
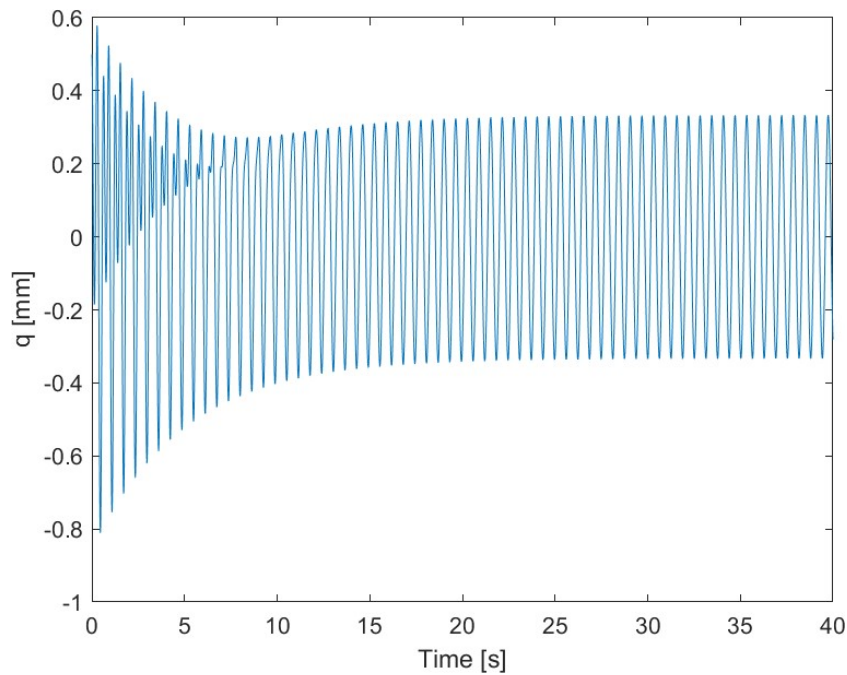
$$\begin{Bmatrix} \dot{q} \\ \ddot{q} \end{Bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} + \begin{bmatrix} 0 \\ M^{-1}b \end{bmatrix} \{u(t)\}$$
$$\{y(t)\} = \begin{bmatrix} c & 0 \end{bmatrix} \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix}$$

https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.solve_ivp.html
Matlab : ode45, ...

To be discussed

- DOF / State / Initial condition
- Newmark, Runge Kutta, ...

Forced harmonic/transient



1 DOF frequency domain / transfer

Dynamic equation

$$\operatorname{Re} \left((-\omega^2 m + i\omega c + k) q(\omega) e^{i\omega t} - F(\omega) e^{i\omega t} \right) = 0$$

Transfer function

$$H(\omega) = \frac{q(\omega)}{F(\omega)} = \frac{1}{-\omega^2 m + i\omega c + k}$$

Fourier / Laplace transform

$$\mathcal{F}(y) = Y(\omega) = \int_{-\infty}^{+\infty} y(t) e^{-i\omega t} dt$$

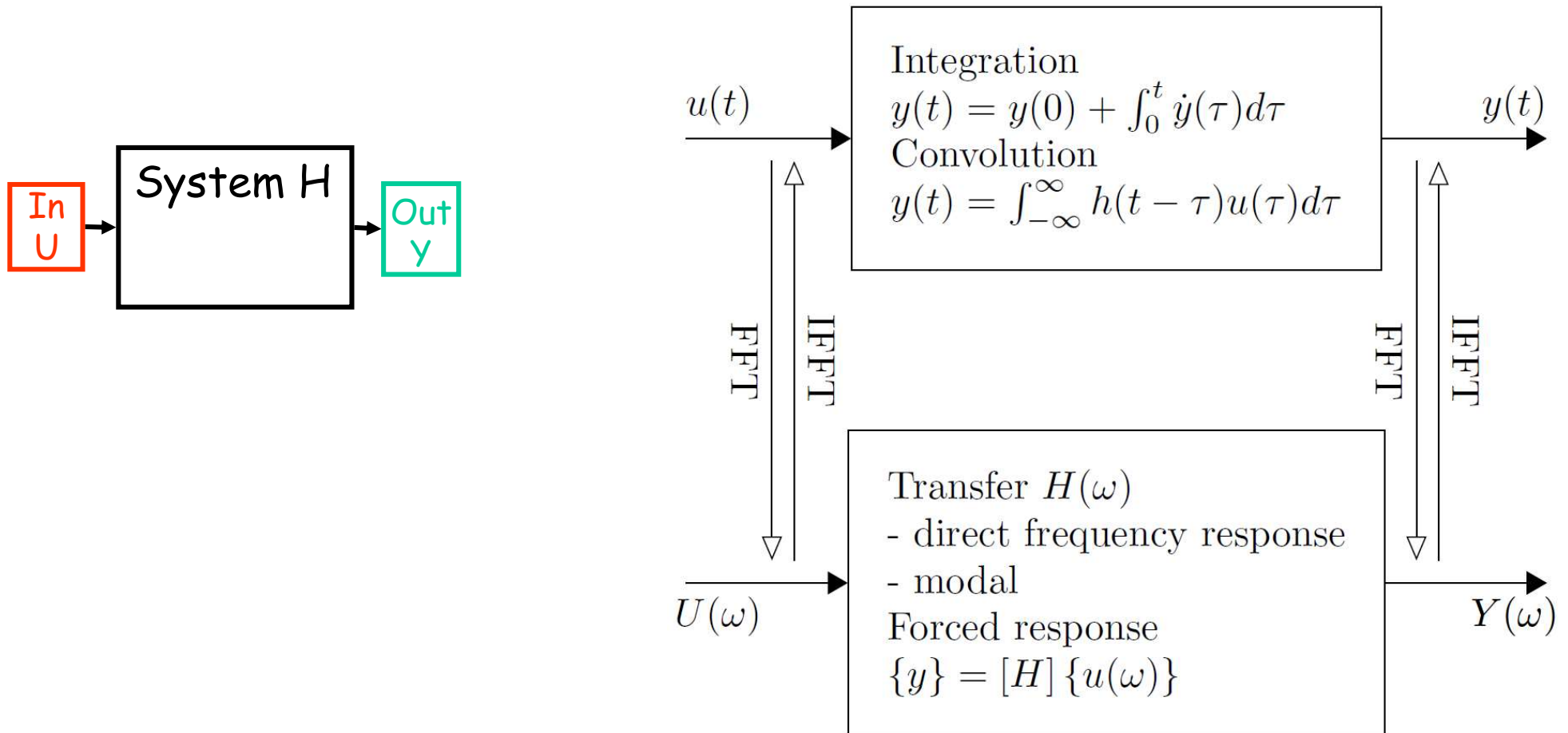
$$\mathcal{F}(\dot{y}) = i\omega \mathcal{F}(y)$$

$$Y(s) = \int_0^{+\infty} y(t) e^{-st} dt$$

$$H(s) = \frac{q}{F} = \frac{1}{ms^2 + cs + k}$$

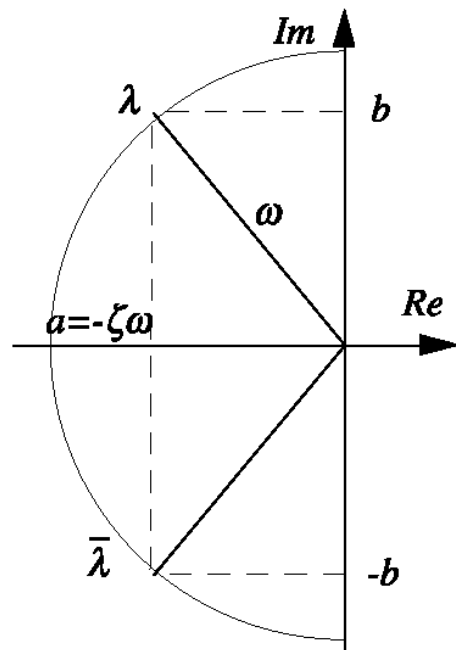
$s = i\omega$ (in France s often noted p)
Laplace/Fourier

Non parametric frequency / time models



Transfer = assume linear time invariant LTI
Non LTI = LPV (linear parameter varying), NL (non-linear)

1 DOF (Bode plot)



1 % damping

$$H(s) = \frac{1}{s^2 m + cs + k} = \frac{1}{m} \left(\frac{\beta}{s - \lambda} + \frac{\bar{\beta}}{s - \bar{\lambda}} \right)$$

Poles

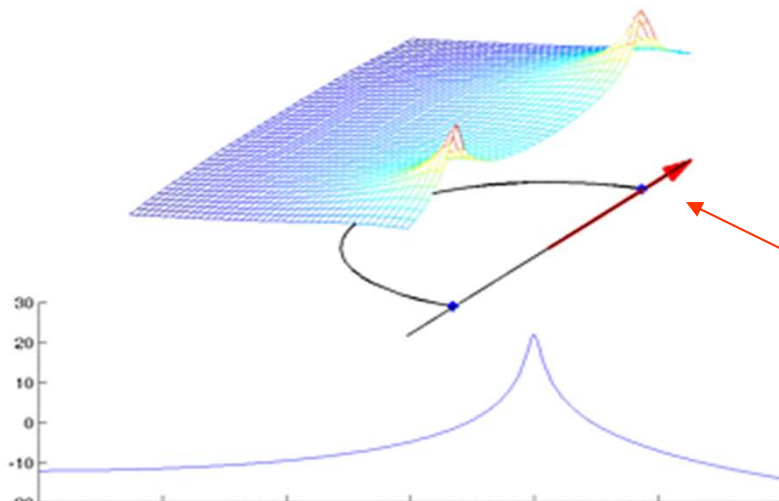
$$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}, \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\omega_n = \sqrt{k/m} = |\lambda|, \quad \zeta = \frac{c}{c_{crit}} = \frac{c}{2\sqrt{km}} = \frac{-\text{Re}(\lambda)}{|\lambda|}$$

1 DOF system (single mode for mechanical system)
has

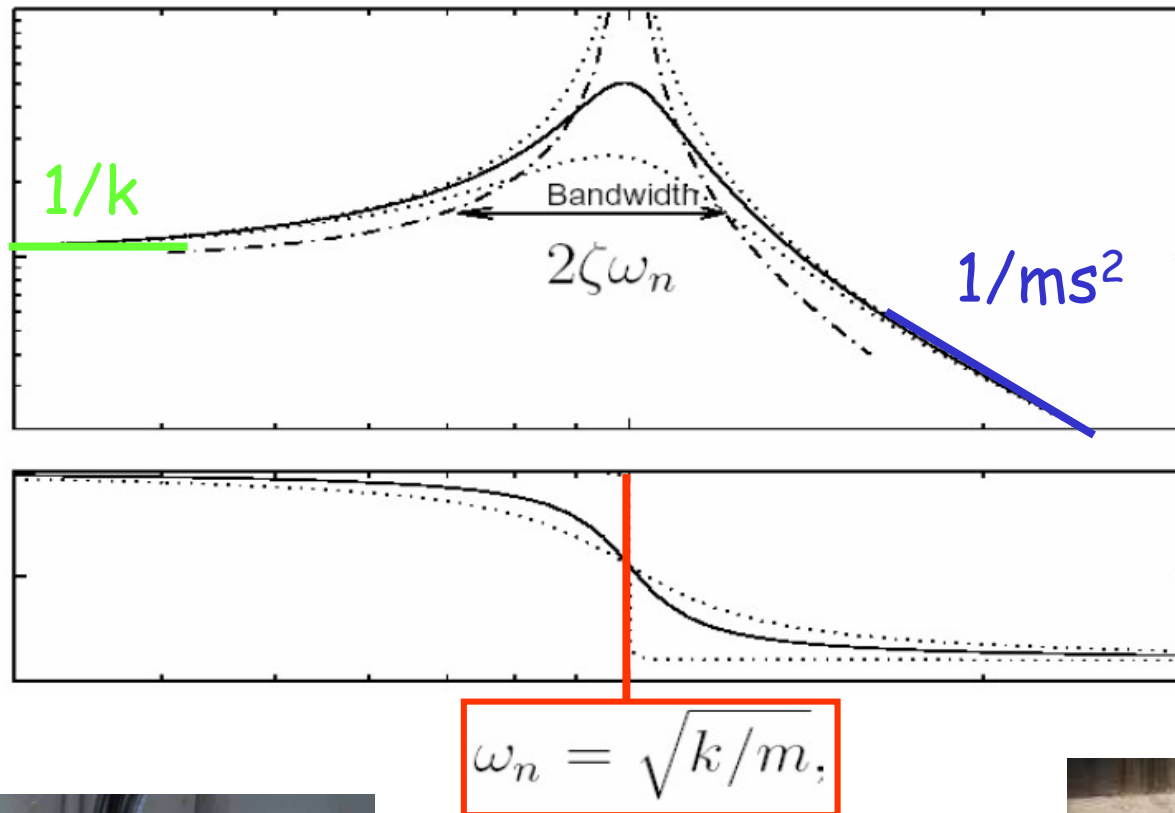
2 complex conjugate poles (linear system modes)

Damping ratio see also quality slide



$s = i\omega$
Laplace/Fourier

1 DOF :Bode plot



1 DOF :Bode plot

$$H(\omega) = \frac{1}{-\omega^2 m + i\omega c + k}$$

Asymptotes:

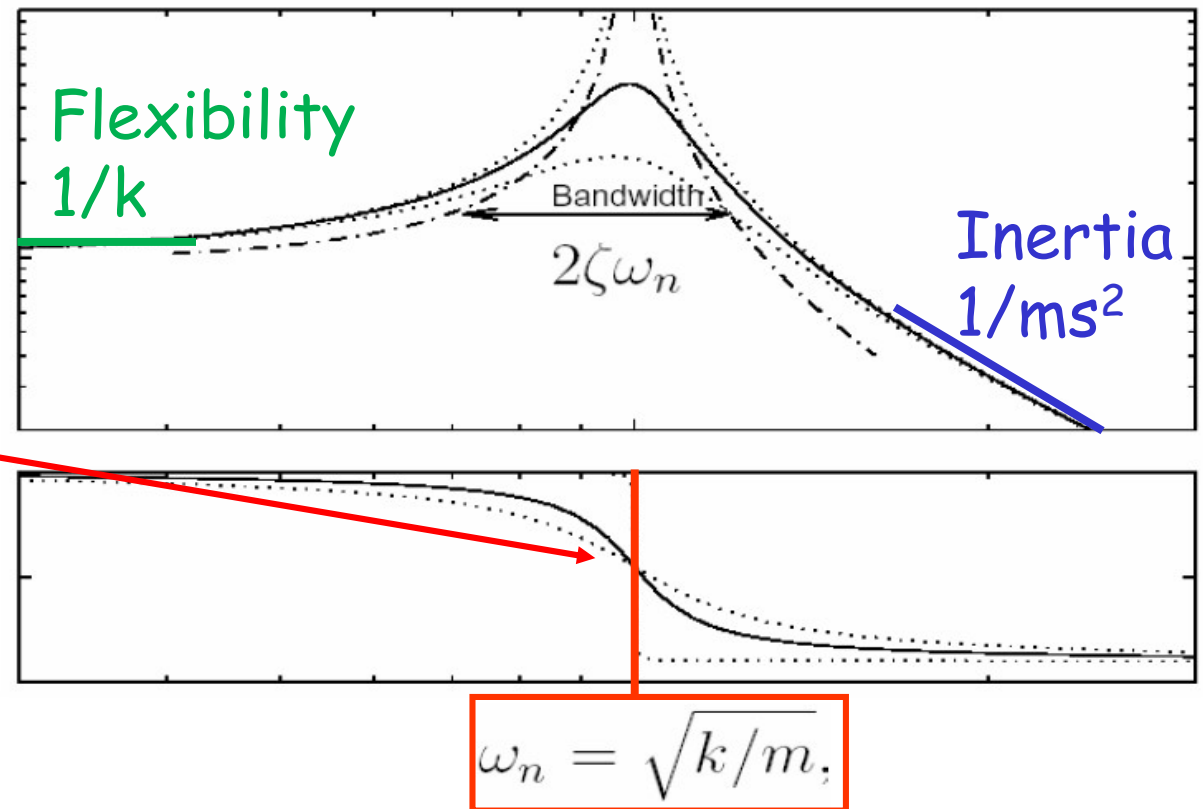
- Flexibility $1/k$
- Inertia (isolation) $1/m\omega^2$

Resonance

- Amplitude $\propto 1/\zeta$
- Phase resonance -90°
- Bandwidth $\propto \zeta$

Response at phase resonance

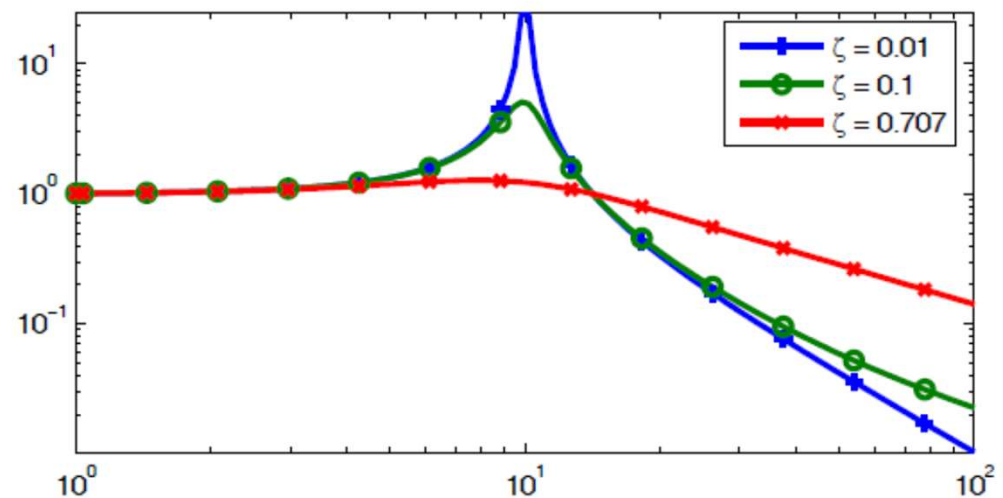
$$H(\omega_n) = 1/i2\zeta\omega_n^2$$



Quality factor

- Other physics definition : $Q = 2\pi \frac{E_{max}}{E_{dis}} = \frac{1}{2\zeta_j}$
- Transmissibility

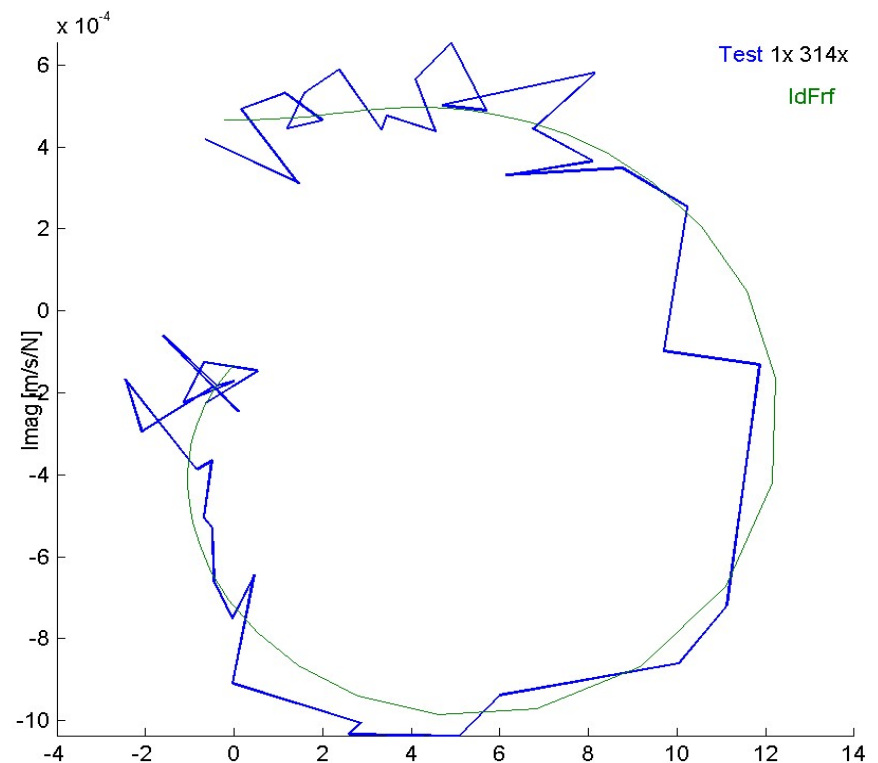
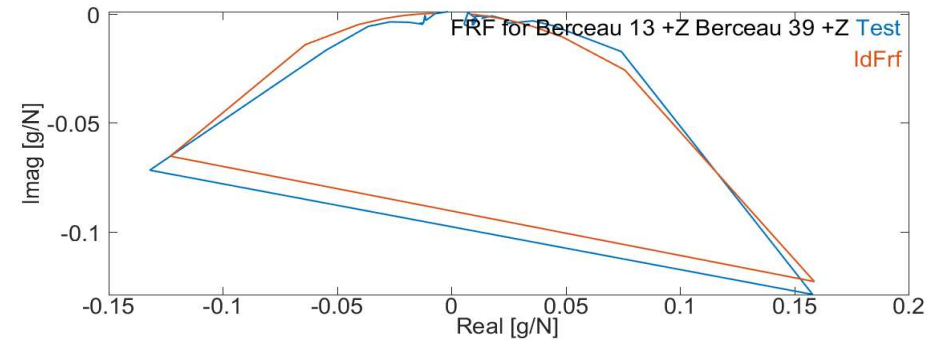
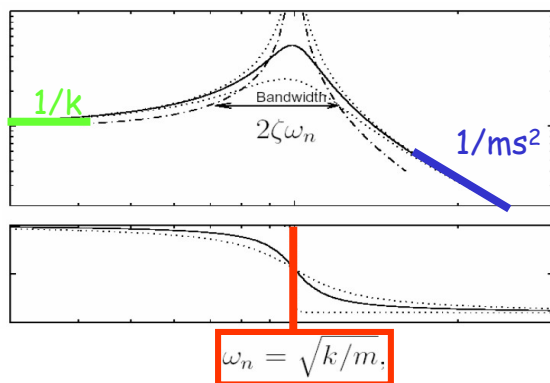
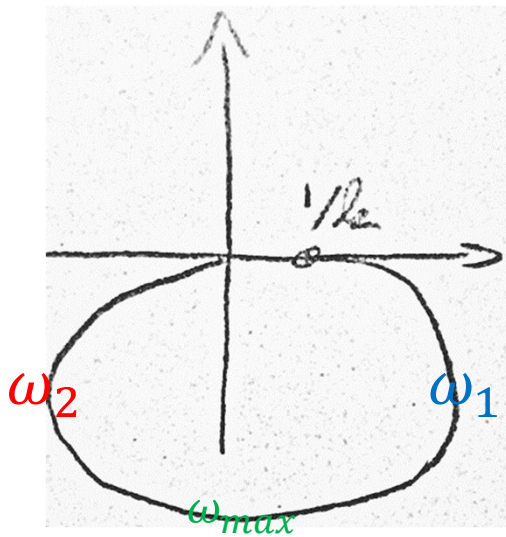
$$H = \frac{x_M}{x_B} = \left[\frac{s^2}{s^2 + 2\zeta_j \omega_j s + \omega_j^2} \right]_{\omega_j} = \frac{1}{2i\zeta_j} = Q$$



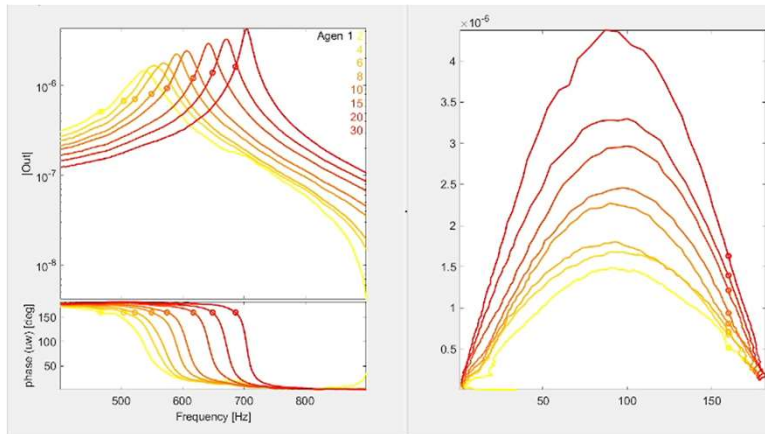
Nyquist & -3dB method

Failures : resolution, noise, multi-mode

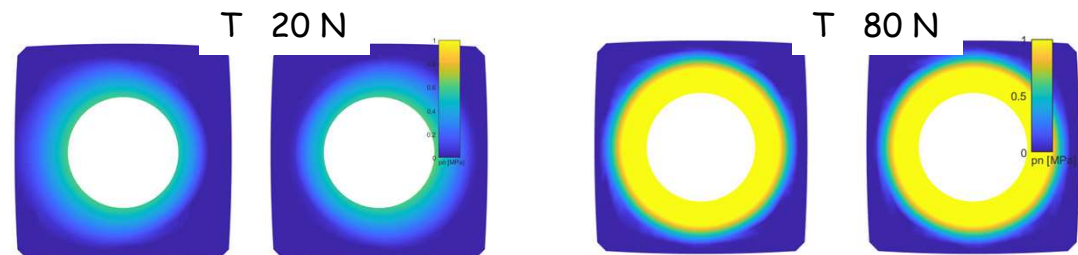
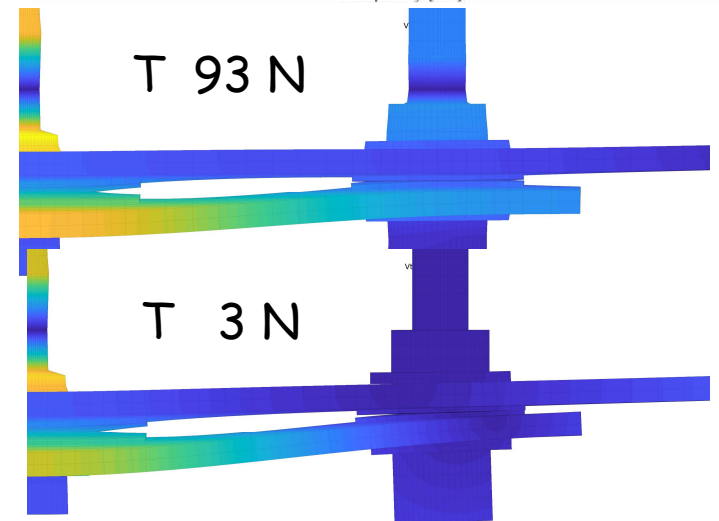
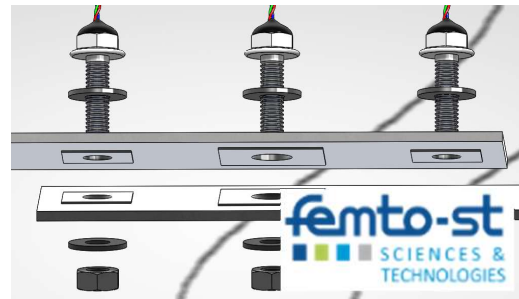
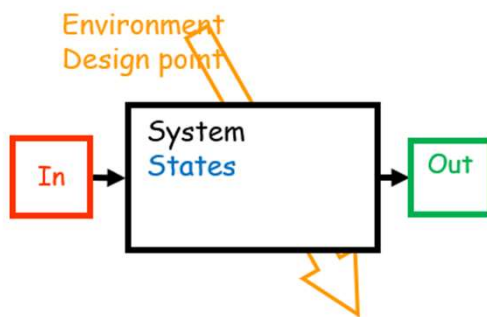
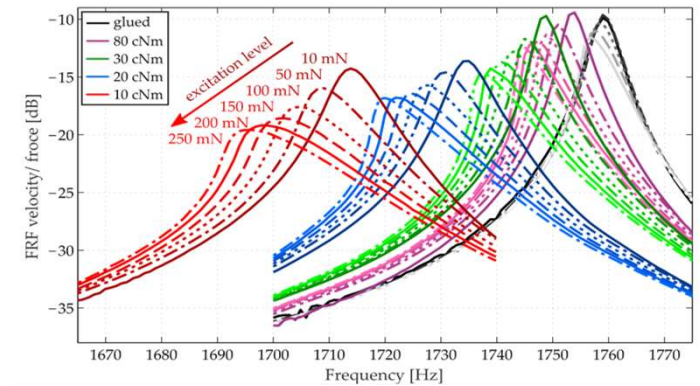
$$\zeta = \frac{\omega_2 - \omega_1}{2\omega_{max}}$$



Resonance of NL / parametric system



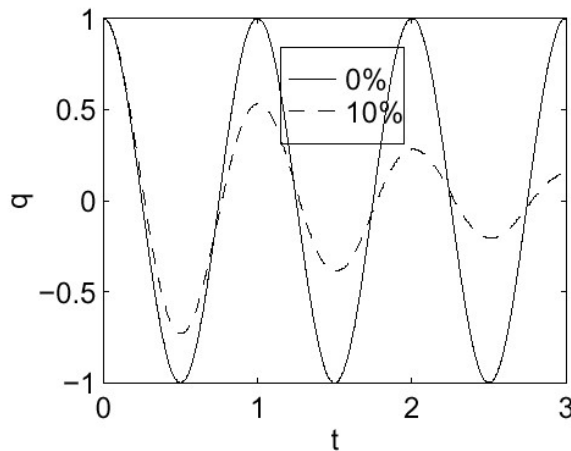
HITACHI
Inspire the Next



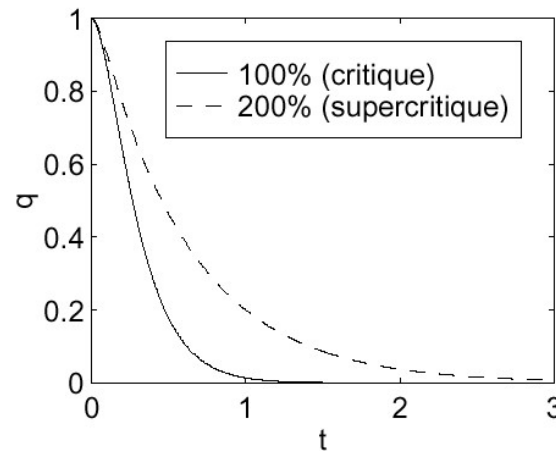
1 DOF : time response / poles

$$q(t) = \operatorname{Re}(Ae^{\lambda_1 t} + Be^{\lambda_2 t}) = A \cos\left(\omega_j \sqrt{1 - \zeta_j^2} t + \phi\right) e^{-\zeta_j \omega_j t}$$

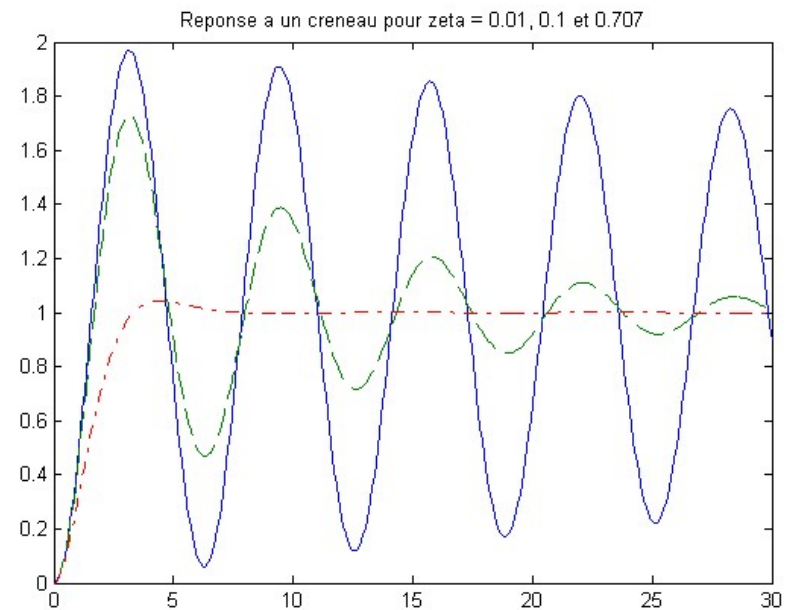
$$\mathcal{L}^{-1}\left(\frac{1}{s - \lambda_1}\right) = e^{\lambda_1 t}$$



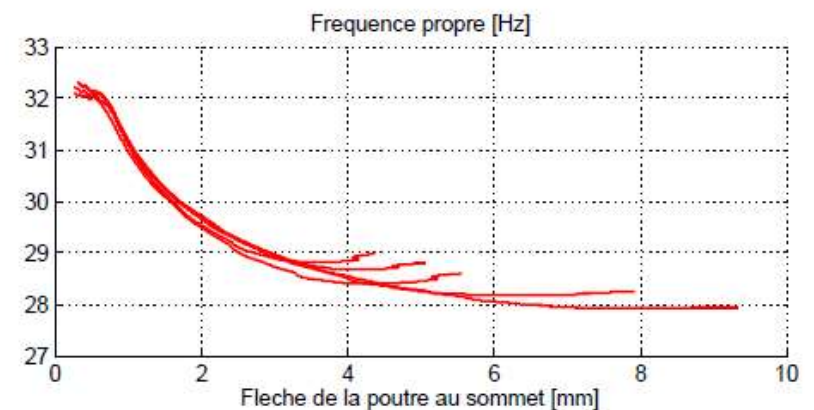
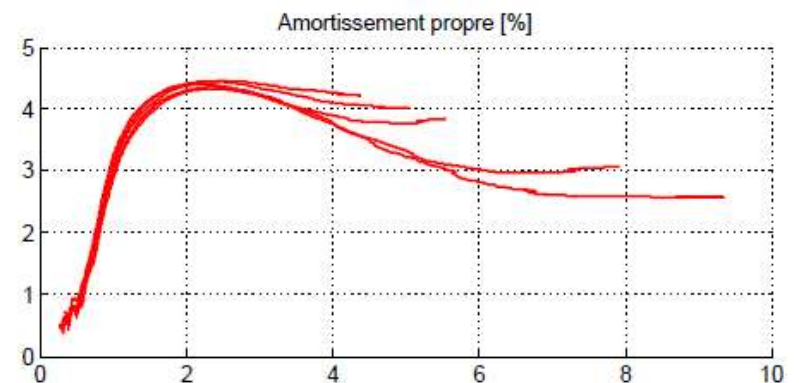
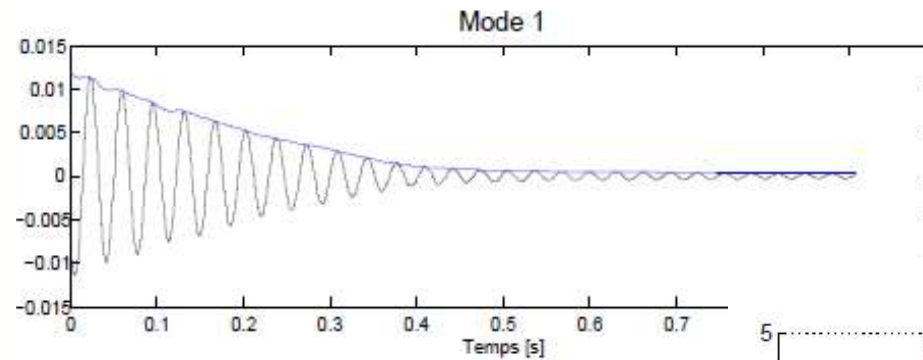
Initial condition



Step input



Logarithmic decrement, evolutions

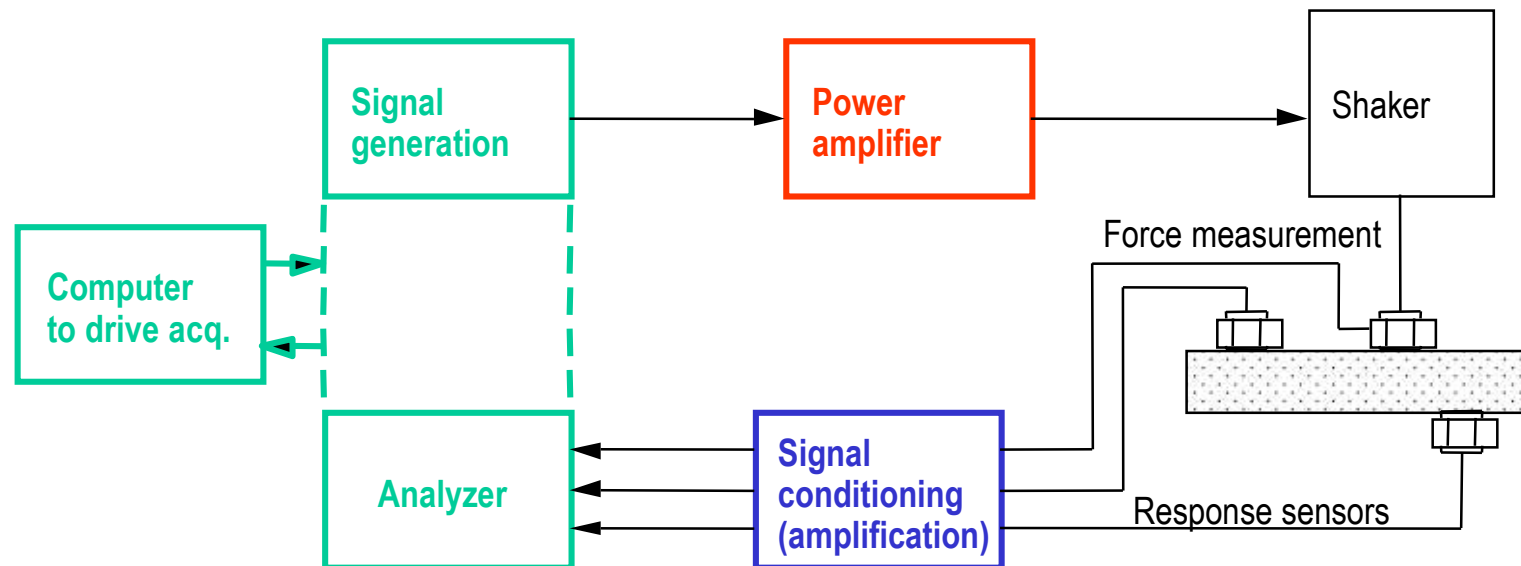
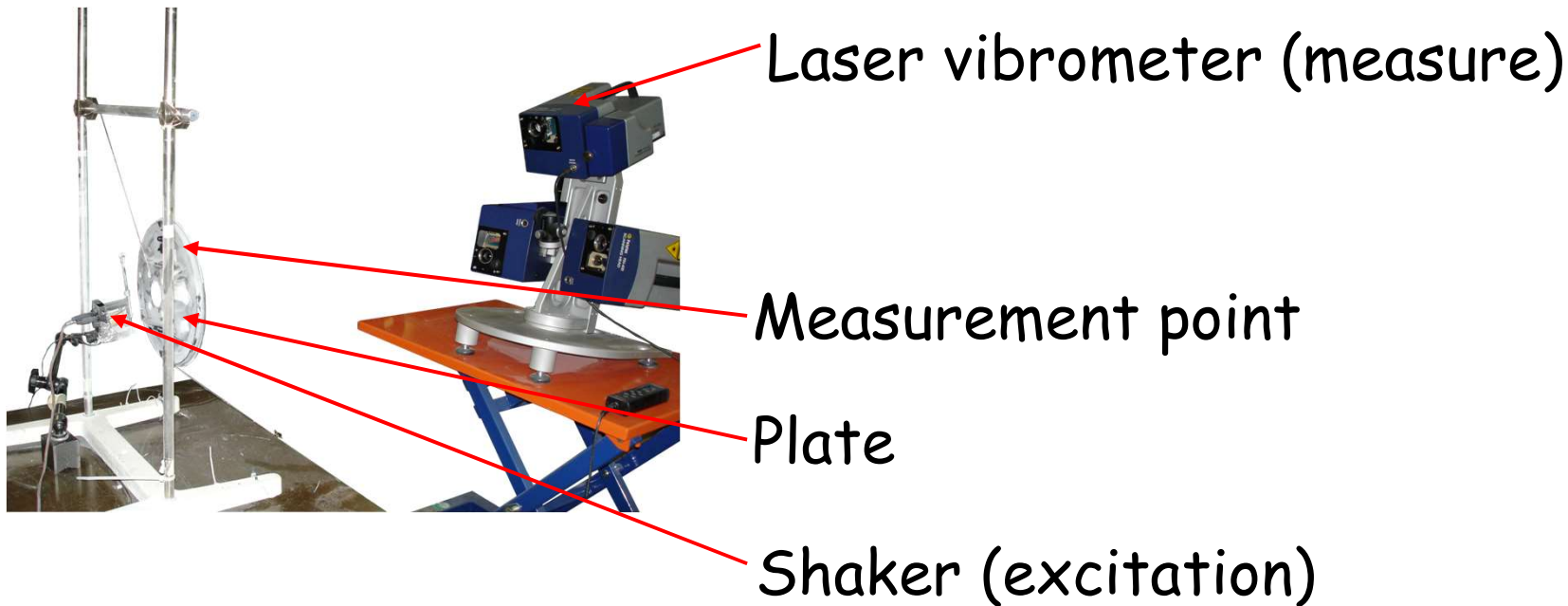


- For non-linear systems ω & ζ are amplitude dependent
- Hilbert transform, Kalman filt., analytic signal
- Sample application :
NES (non-linear energy sink)

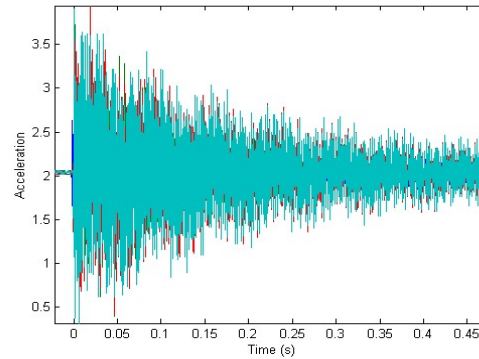
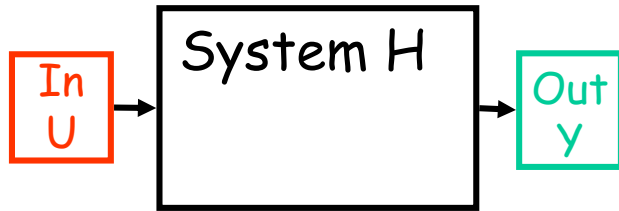
Modes & synthesis (course 3, see also second part of [Modal.pdf](#) slides)

1. Inputs/outputs, IO shape matrix, disp, resultants, ... ([s2.1](#))
2. Discrete modes (harmonic sol. without input), orthogonality ([section 2.2.1](#))
3. Ritz/Galerkin principles
4. Ritz/modal coordinates, PPV, series & state-space, time/freq strategies
5. Modal & **Rayleigh damping**, modal damping in physical coord ([s 2.2.x](#))
6. Peak visibility, truncation, effective contributions ([s 2.2.x](#))

Experimental modal analysis : measurements



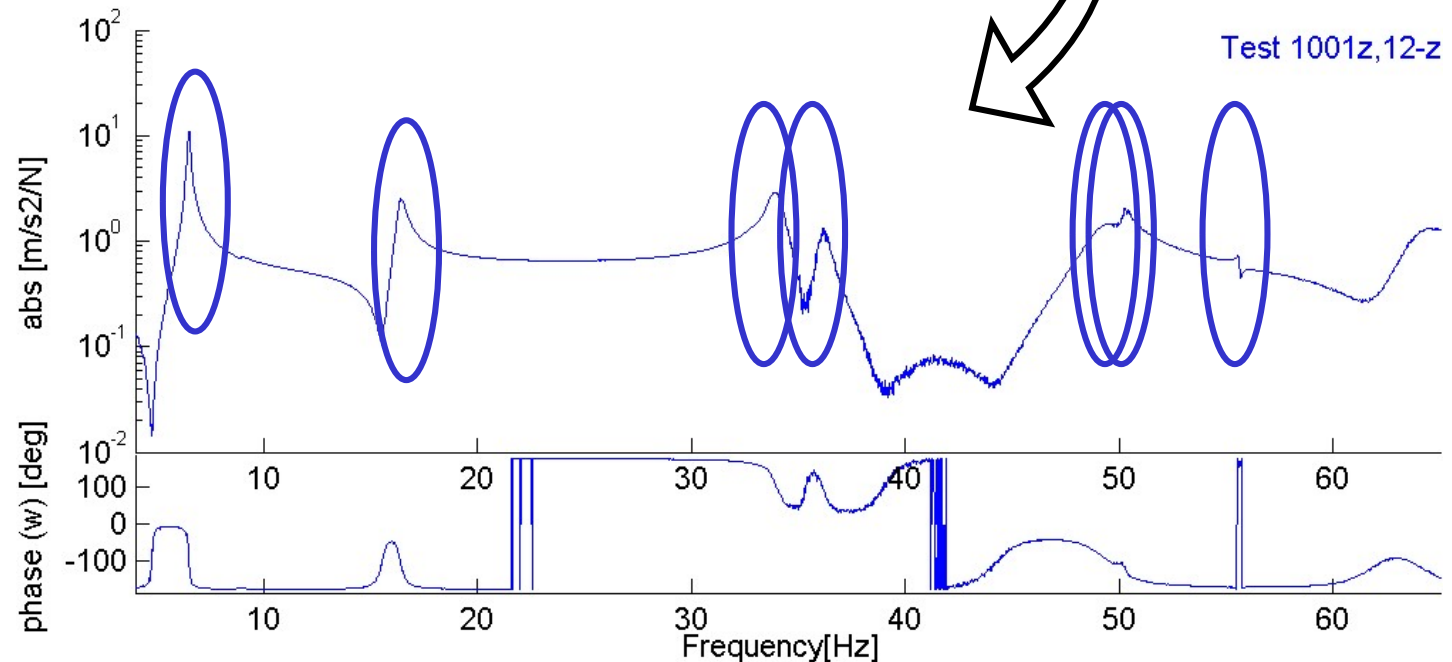
Modal analysis : transfers



Transfers estimated from time response

ONE input
ONE output

$$\{Y(\omega)\} = [H(\omega)]\{U(\omega)\}$$

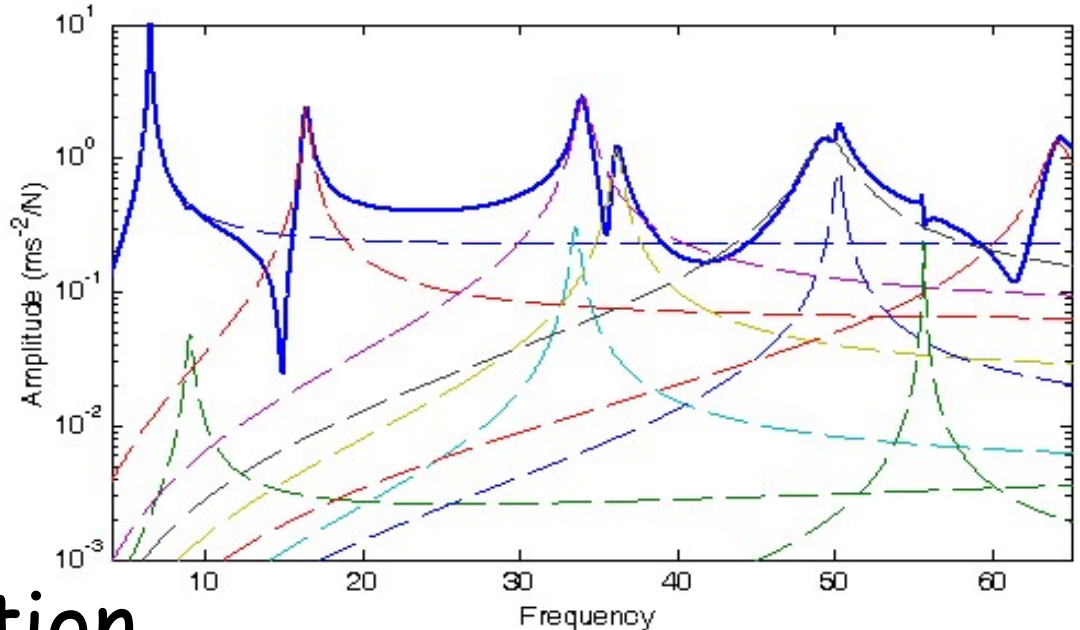


MANY resonances

Bode plot : visualization of transfer function

1 input, 1 output, many resonances

MDOF multiple degree of freedom
SISO single input single output



Spectral decomposition

MDOF (multiple resonances)

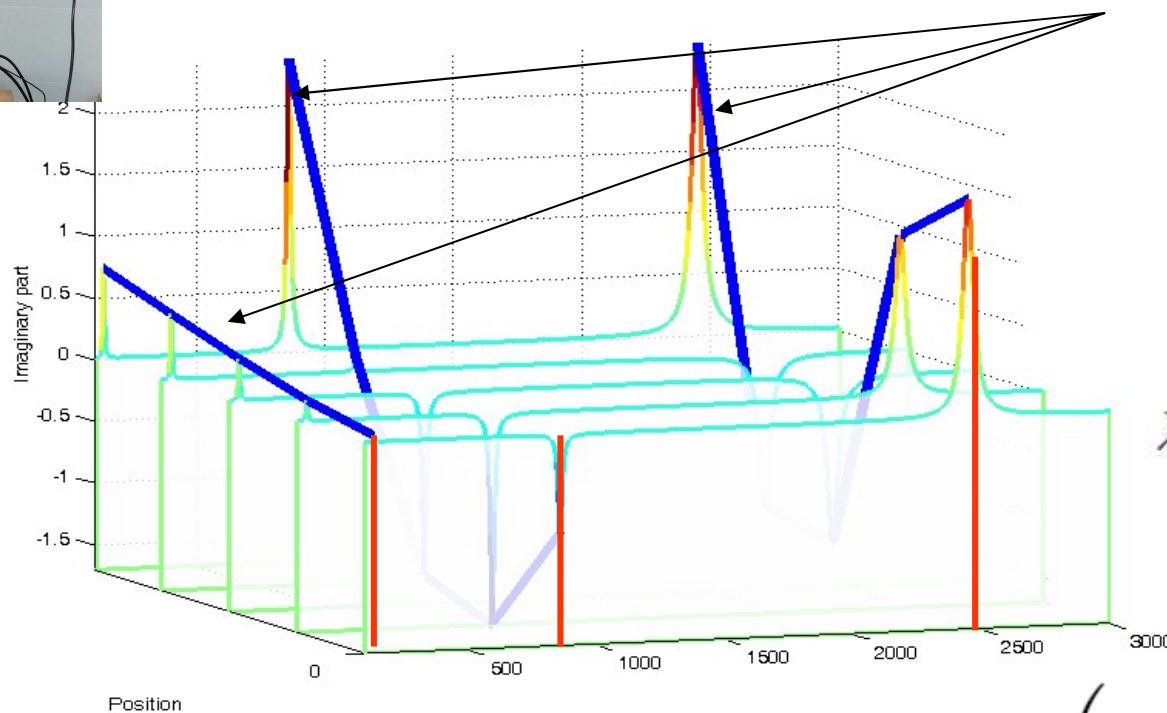
SISO T_j is 1x1

$$[\alpha(s)] = \sum_{j \in \text{identified}} \left(\frac{[T_j]}{s^2 + 2\zeta_j \omega_j s + \omega_j^2} \right)$$

Note : series truncated in practice

Constant approximation of high frequencies (D term in states space models)

MDOF MIMO system



The shapes

The poles

$$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

$$[\alpha(s)] = \sum_{j \in \text{identified}} \left(\frac{[T_j]}{s^2 + 2\zeta_j\omega_j s + \omega_j^2} \right)$$

- Poles depend on the system (not the input/output)
- The shape is associated with the input/output locations

Lagrange equations / virtual power principle

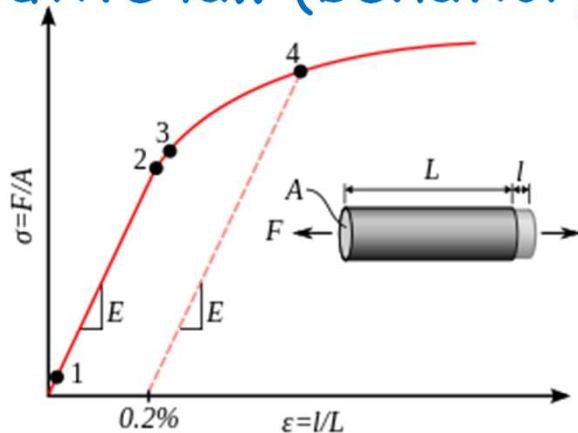
Kinematics

- Displacement $u(x, t)$
- Strain $\epsilon(x, t)$

Statics/thermodynamics

- Load/Stress $\sigma(x, t)$
- Power : $\int_{\Omega} \sigma \dot{\epsilon}$

Constitutive law (behavior)



LTI Equations of motion

$$[M] \{\ddot{q}(t)\} + [C] \{\dot{q}(t)\} + [K] \{q\} = \{F(t)\}$$

Modes : **harmonic** solution with **no force**

$$[Ms^2 + Cs + K]\{q(s)\} - \{F(s)\} = 0$$

$q(t) = \text{Re}(\{\psi_j\}e^{\lambda_j t})$ complex mode (general definition)

$$\text{Re}([M\lambda_j^2 + C\lambda_j + K]\{\psi_j\}e^{\lambda_j t} - \{0\}) = 0$$

Eigenvalue problem

Linear time invariant (LTI)



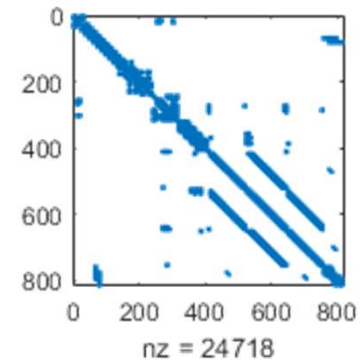
Normal modes of elastic structure

- Real mode : no damping / elastic / conservative

$$q(t) = \text{Re}(\{\phi_j\}e^{i\omega_j t})$$

$$[K - \omega_j^2 M]_{N \times N} \{\phi_j\}_{N \times 1} = \{0\}$$

- $M > 0$ & $K \geq 0 \Rightarrow \phi$ real
- There are N distinct modes for N DOF
- Full solver : `scipy.linalg.eig` (LAPACK Linear Algebra)
- Partial solvers exist, a few keywords
 - `scipy.sparse.linalg.eigs` (Matlab eigs) : Arnoldi
 - FEM Solvers : `Lanczos` (Krylov+conjugate gradient), `AMLS`



Normal modes of elastic structure

- Orthogonality

$$\{\phi_k\}^T [K - \omega_j^2 M] \{\phi_j\} = \{0\}$$

$$\{\phi_j\}^T [K - \omega_k^2 M] \{\phi_k\} = \{0\}$$

$$\phi_j^T M \phi_k = \mu_j \delta_{jk}$$

$$\phi_j^T K \phi_k = \mu_j \omega_j^2 \delta_{jk}$$

- Scaling conditions

- Unit mass

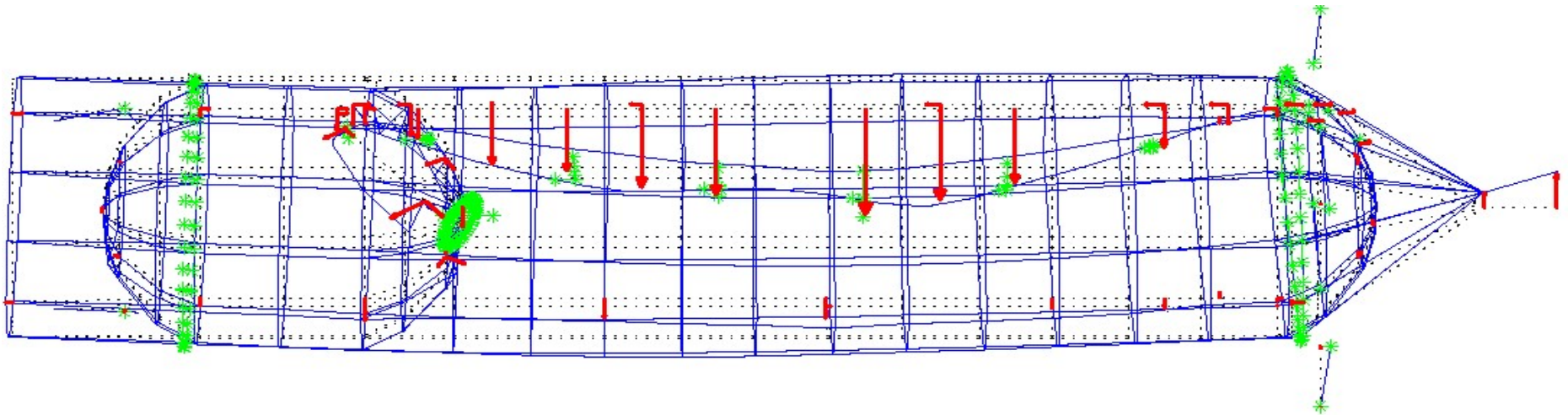
$$\{\phi_j\}^T [M] \{\phi_j\} = 1$$

- Unit amplitude

$$[c_s] \{ \tilde{\phi}_j \} = 1 \quad \mu_j(c_s) = ([c_i] \{ \phi_j \})^{-2}$$

normalized at s

mass normalized

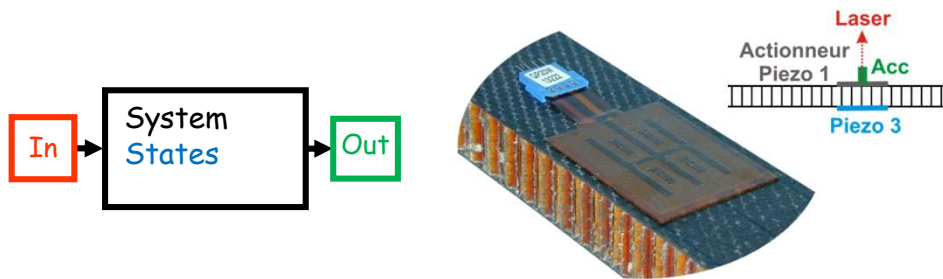


Kinematics / model reduction

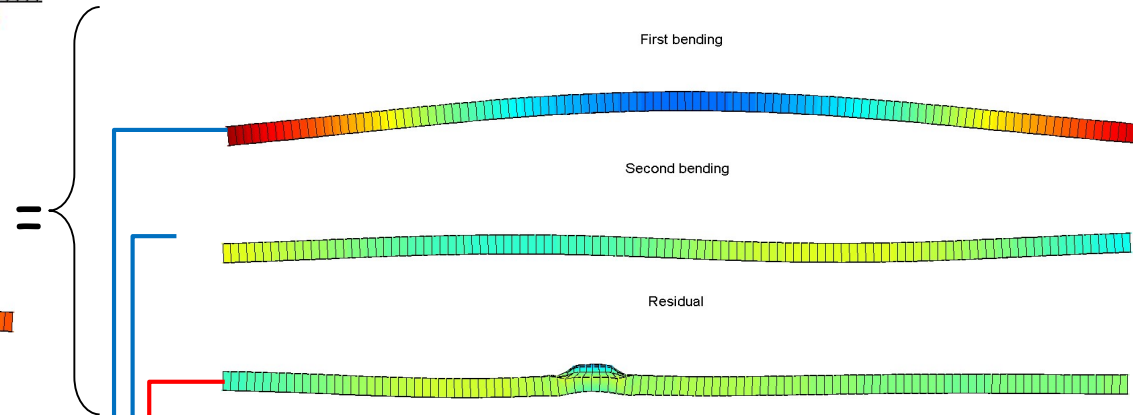
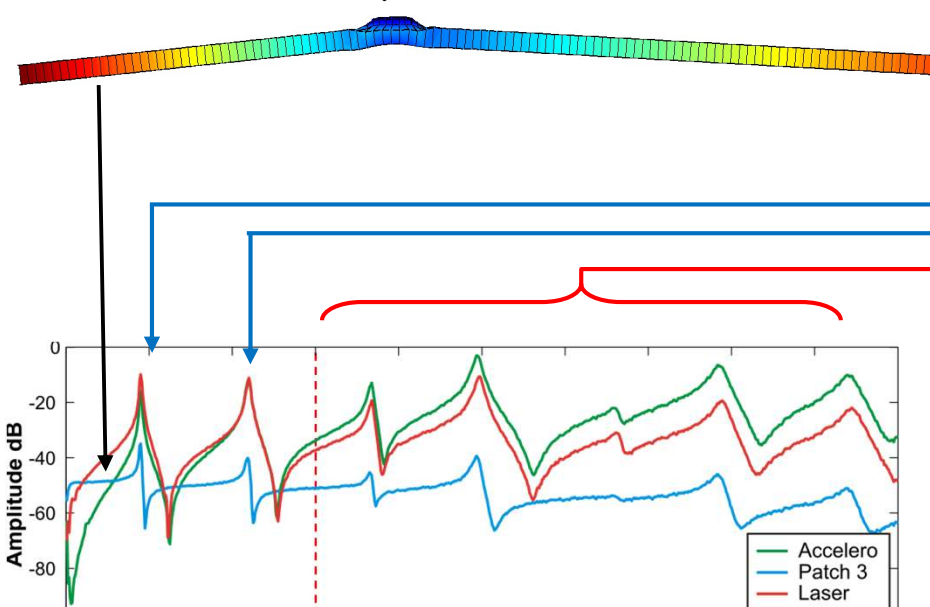
- Displacement $u(x,t)$
 $= \text{shapes}(x) \times \text{DOF}(t)$

$$\{q\}_N = \begin{bmatrix} \text{Shapes} & \text{DOFs} \end{bmatrix} = \begin{bmatrix} T & q_R \end{bmatrix}$$

$N \times N_R$



Quasi-static response @ 10Hz



- Modes** : high energy, load independent (no blister shape)
- Static** response (influence of input=blister), important away from resonance

Modal (principal) coordinates

- **Coordinate change** (physical q , generalized q_R):

$$\{q(s)\} = [T]_{Nq \times Nqr} \{q_R(s)\} = [\phi_1 \dots \phi_{NM}] \{q_R(s)\}$$
- Inject in equations of motion

$$[Ms^2 + Cs + K]_{Nq \times Nq} [T] \{q_R(s)\}_{Nqr \times 1} = \{F(s)\}_{Nq \times 1}$$
- Over determined ($N_q \gg N_{qr}$) : compute "virtual work"

$$[T]^T [Ms^2 + Cs + K] [T] \{q_R(s)\} = [T]^T \{F(s)\}$$
- For modes $[T] = [\phi_1 \dots \phi_{NM}]$

Reduced mass

Reduced stiffness

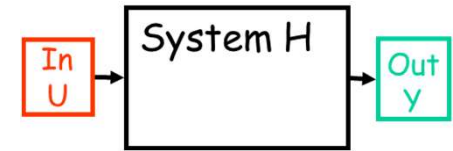
Need : observation, loads/command, damping

$$M_R = T^T M T = [\phi_i^T M \phi_k] = \begin{bmatrix} 1 & \\ & \end{bmatrix} = I$$

$$K_R = T^T K T = \begin{bmatrix} \omega_j^2 & \\ & \end{bmatrix}$$

Observation

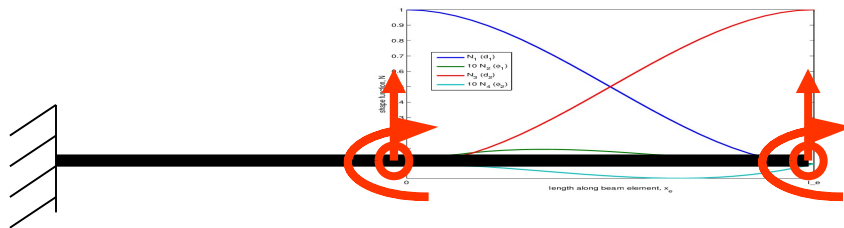
$$\begin{aligned} [Ms^2 + Cs + K] \{q\} &= \{F(s)\} \\ \{y(s)\} &= [c] \{q\} \end{aligned}$$



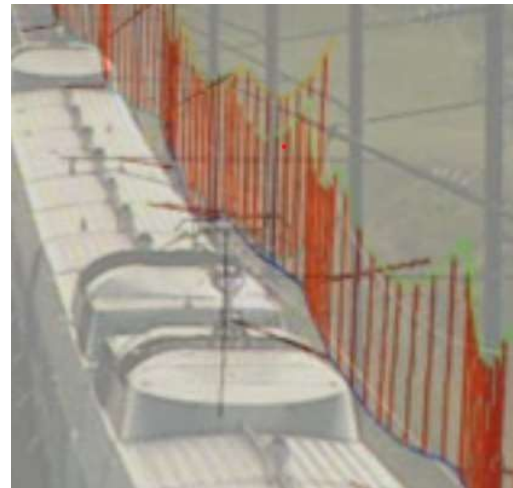
- $\{y\}$ outputs are linearly related to DOFs $\{q\}$ using an observation equation

$$\{y\} = [c] \{q(t)\}$$

- Simple case : extraction $\{w_2\} = [0 \ 0 \ 1 \ 0] \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$

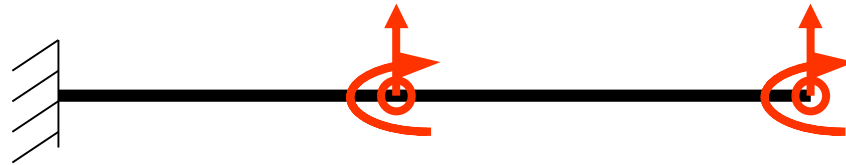


- More general : intermediate points, reactions, strains, stresses, ...



Command

$$[Ms^2 + Cs + K] \{q\} = \{F(x, s)\} = [b(x)]\{u(s)\}$$



- Loads decomposed as spatially unit loads and inputs
 $\{F(t)\} = [b] \{u(t)\}$

Abaqus : *CLOAD + *AMPLITUDE, ...

NASTRAN : FORCE-MOMENT + RLOAD

ANSYS, CODE Aster, ...

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \{F_2\}$$

Modal damping

Rayleigh damping

- Physical domain $[C] = \alpha[M] + \beta[K]$

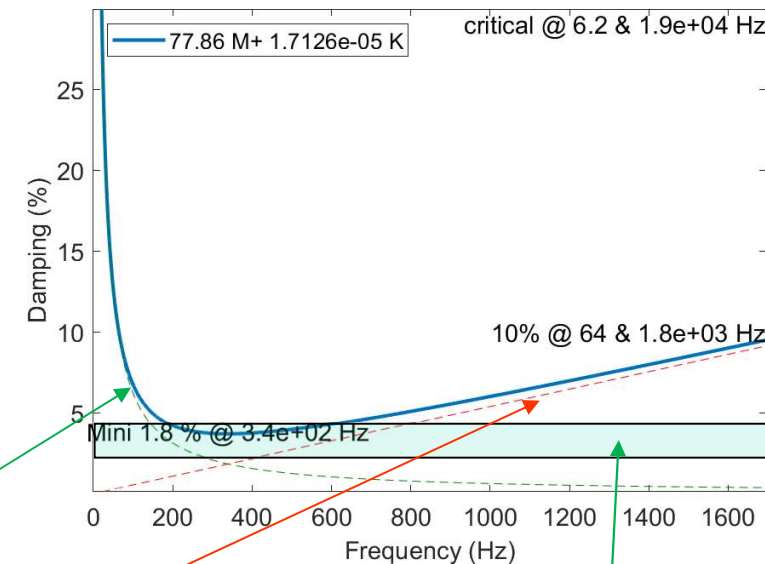
- Modal $\phi^T[C]\phi = [2\zeta_j\omega_j] = \alpha[I] + \beta[\omega_j^2]$

$$\zeta_j = \frac{\alpha}{2\omega_j} + \frac{\beta\omega_j}{2}$$

Mass

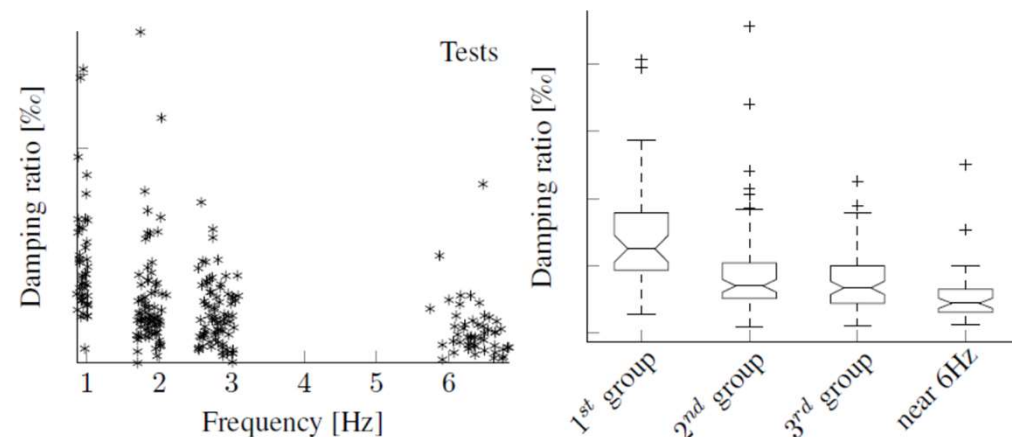
Stiffness
can be > 100%

Reality



Modal damping ζ_j derived from test

Physical domain $[C] = [M\phi] [2\zeta_j\omega_j] [\phi^T M]$



Physical / modal & spectral decomposition

- Physical

$$[Ms^2 + Cs + K] \{q(s)\}_{Nq} = [b] \{u(s)\}$$

$$\{y(s)\} = [c] \{q(s)\}$$



- Modal coordinate

$$\{q(s)\} = [T] \{q_R(s)\} = [\phi_1 \dots \phi_{NM}] \{q_R(s)\}$$

- Modal equations (modal damping)

$$\left[Is^2 + [2\zeta_j \omega_j] s + [\omega_j^2] \right] \{q_R(s)\}_{Nqr} = [\phi_j^T b] \{u(s)\}$$

$$\{y(s)\} = [c \phi_j] \{q_R(s)\}$$

- Reduced matrices = diagonal
Modal observability/commandability

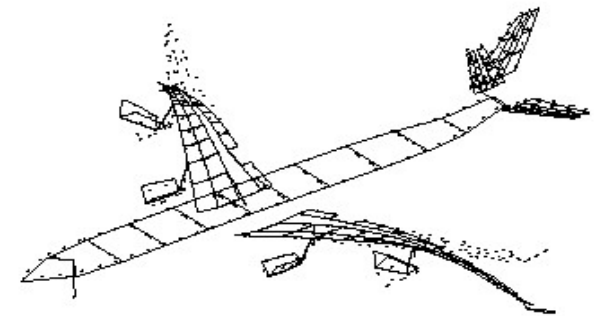
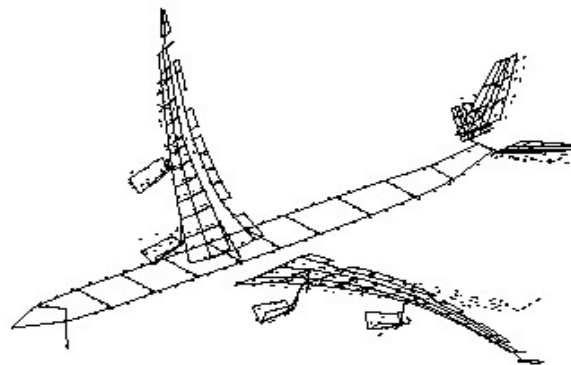
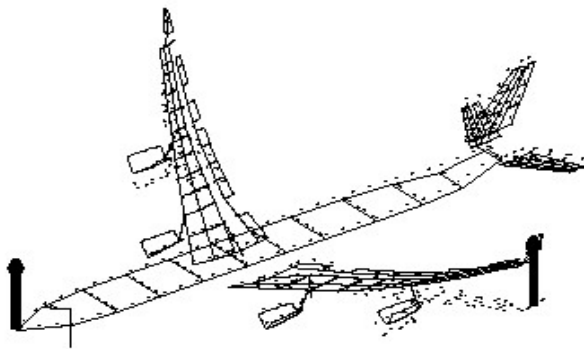
- Spectral equations (inverse of diagonal matrix)

$$H(s) = [c][Ms^2 + Cs + K]^{-1}[b] = \sum_j \frac{\{c \phi_j\} \{\phi_j^T b\}}{s^2 + 2\zeta_j \omega_j s + \omega_j^2}$$



Observability/controlability

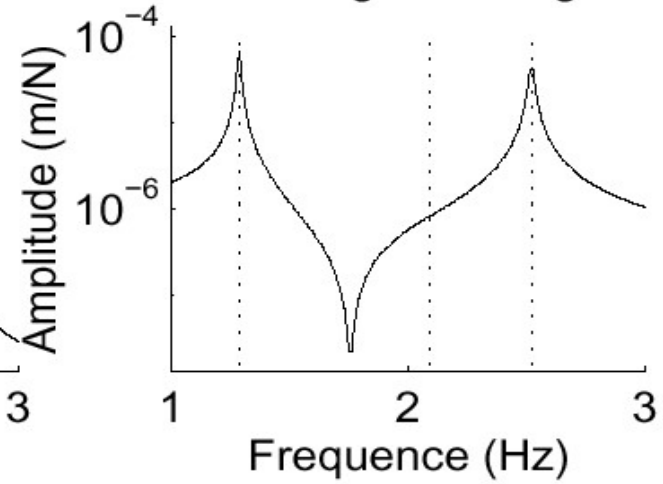
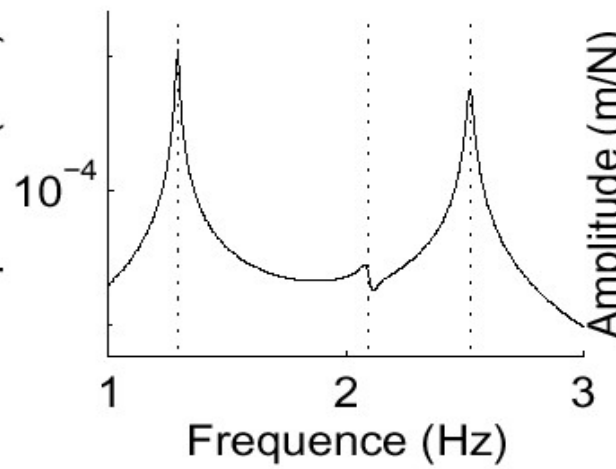
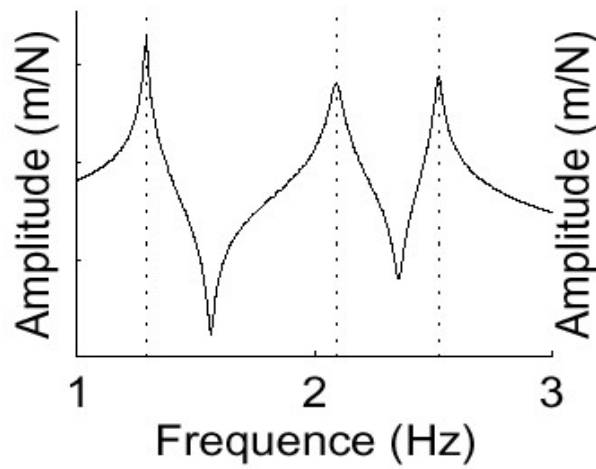
$$H(s) = \sum_{j=1}^N \frac{[c]\{\phi_j\}\{\phi_j\}^T[b]}{s^2 + 2\zeta_j\omega_j s + \omega_j^2} = \sum_{j=1}^N \frac{[T_j]}{s^2 + 2\zeta_j\omega_j s + \omega_j^2}$$



Aile/Aile

Aile/Fuselage

Fuselage/Fuselage



Discretized/reduced equations of motion

FEM $\Leftrightarrow$ Reduction

	Finite elements Continuous $\rightarrow$ discrete full	Reduction Full $\rightarrow$ reduced
Support	Element: line, tria, tetra, ...	FE mesh
Variable separ. Shape functions	$w(x, t) = N_i(x)q_i(t)$ $\epsilon(x, t) = B_i(x)q_i(t)$	$\{q(t)\} = \{T_i\}q_i(t)$ T_i simple FE solutions
Matrix comp. Weak form	$K_{ij} = \int_{\Omega} B_i^T \Lambda B_j = \sum_g B_i^T(g) \Lambda B_j w_g J_g$ numerical integration	$K_{ijR} = T_i^T K T_j$ FEM matrix projection
Assembly	Localization matrix	Boundary continuity, CMS
Validity	Fine mesh for solution gradients	Good basis for considered loading (bandwidth/inputs)

[1] O. C. Zienkiewicz et R. L. Taylor, *The Finite Element Method*. MacGraw-Hill, 1989

[2] J. L. Batoz et G. Dhatt, *Modélisation des Structures par Éléments Finis*. Hermès, Paris, 1990

[3] K. J. Bathe, *Finite Element Procedures in Engineering Analysis*. Prentice-Hall Inc., Englewood Cliffs, NJ, 1982

LTI model forms

- See MATLAB control or scipy.signal.lti

- state-space $\begin{aligned} \{\dot{x}\} &= [A]\{x\} + [B]\{u\} \\ \{y\} &= [c]\{x\} + D\{u\} \end{aligned}$

- Mechanical SS
$$\begin{aligned} \begin{Bmatrix} \dot{p} \\ \ddot{p} \end{Bmatrix} &= \begin{bmatrix} 0 & I \\ -[\omega_j^2] & -\Gamma \end{bmatrix} \begin{Bmatrix} p \\ \dot{p} \end{Bmatrix} + \begin{bmatrix} 0 \\ \phi_j^T b \end{bmatrix} \{u(t)\} \\ \{y(t)\} &= [c\phi_j \quad 0] \begin{Bmatrix} p \\ \dot{p} \end{Bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{Bmatrix} \dot{q} \\ \ddot{q} \end{Bmatrix} &= \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} + \begin{bmatrix} 0 \\ M^{-1}b \end{bmatrix} \{u(t)\} \\ \{y(t)\} &= [c \quad 0] \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} \end{aligned}$$

- tf $H = \sum_i b_i s^i / \sum_j a_j s^j$, zpk (zero pole gain), residue (partial fraction expansion = modal form)

Complex mode to real state-space

If non modal damping, complex conjugate poles

$$[M\lambda_j^2 + C\lambda_j + K]\{\psi_j\} = 0 \quad \left(\begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \lambda_j + \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \right) \{\theta_j\} = \{0\}$$

$$\{\theta_j\} = \begin{Bmatrix} \psi_j \\ \psi_j \lambda_j \end{Bmatrix}$$

can be grouped as in elastic mode state-space

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & I \\ -\omega_j^2 & -2\zeta_j \omega_j \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} \{u(t)\}$$

$$\{y(t)\} = [C_1 \ C_2] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$\begin{Bmatrix} B_{j1} \\ B_{j2} \end{Bmatrix} = 2 \begin{bmatrix} \operatorname{Re}(\theta_j^T B) & \operatorname{Im}(\theta_j^T B) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \zeta_j \omega_j & -\omega_j \sqrt{1 - \zeta_j^2} \end{bmatrix}$$

$$\begin{Bmatrix} C_{1j} \\ C_{2j} \end{Bmatrix} = \frac{1}{\omega_j \sqrt{1 - \zeta_j^2}} \begin{bmatrix} \omega_j \sqrt{1 - \zeta_j^2} & \zeta_j \omega_j \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \operatorname{Re}(C \theta_j) \\ \operatorname{Im}(C \theta_j) \end{bmatrix}$$

Difference : loads affect velocity $B_1 \neq 0$ and $C_2 \neq 0$