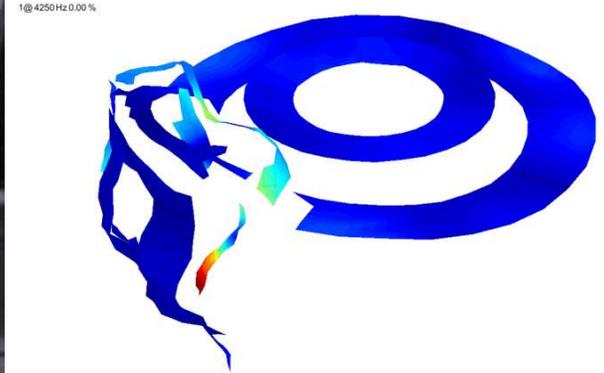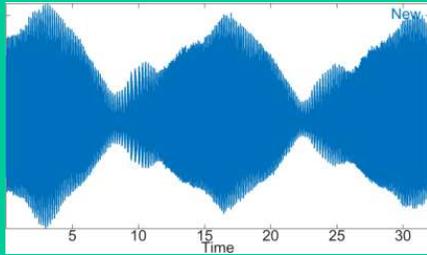# SVD, Advanced modal

1. Squeal motivation : Even in fairly complex case « the response occurs with a restricted subspace » = shapes remain invariant in some sense

2. Subspace building strategies :
   - CMS : modes + static + parameters
   - Time & frequency snapshots, wave/cyclic computations, …

3. Basis building strategies
   - Gram-Shmidt / LU : classical non-sorted
   - Sorting contributions : SVD
   - Choosing norms (SVD variants)
   - Analyzing right singular vectors (modal/generalized coordinates)
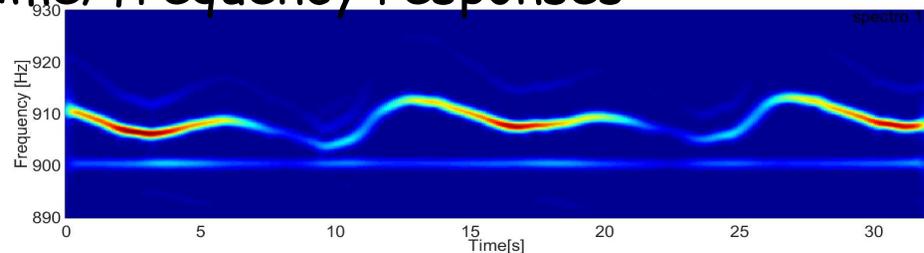
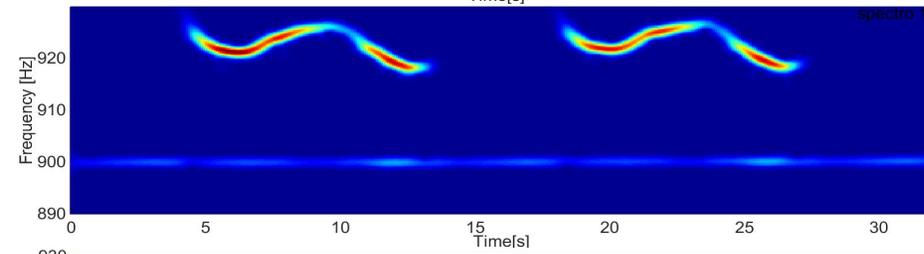   - Other classic use : least squares & conditioning

SD Tools
VIBRATION SOFTWARE & CONSULTING

Arts et Métiers
Institute of Technology

MS2SC   http://savoir.ensam.eu/moodle/course/view.php?id=1874
PROVIR   http://savoir.ensam.eu/moodle/course/view.php?id=490

Learning shapes in squeal event

Time measurement during squeal

Sample time/frequency responses

Variability
- influence of wheel angle

Reproductibility
- Multiple events

Braking event 1

Braking event 2

Braking event 3

What is expected from theory?

Shape 1 : real mode with **normal** contribution

Mode 8 at 5047 Hz 0.03 %

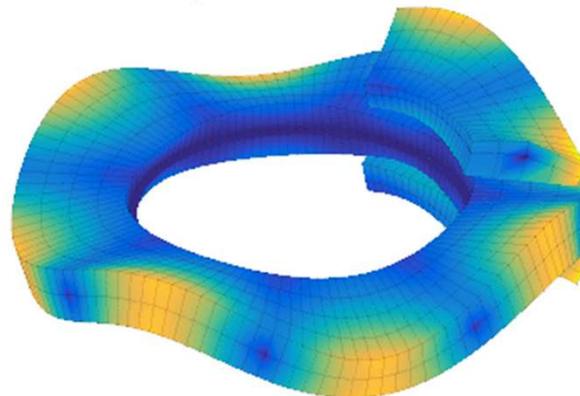$$\sum_i \left\{ \begin{array}{c} U_i \\ \downarrow \\ sensors \end{array} \right\} \{a_i(t) \quad \rightarrow \quad time\}$$

2 DOF : amplitudes of each shape
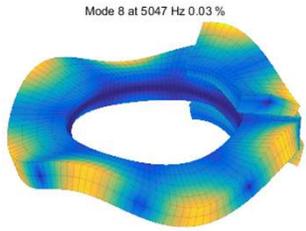
Shape 2 : real mode with **tangential** contribution

Mode 9 at 5058 Hz 0.08 %

Parameter : friction = non symmetric coupling

normal displacement $\Rightarrow$ tangential load

$$[K_u] = [\boldsymbol{b_{TAN}}] \left[\begin{array}{c} \backslash \mu w_j J \frac{\partial p}{\partial q} \backslash \end{array} \right] [\boldsymbol{c_{NOR}}]$$

μ=0.1 at 5052 Hz -0.13 %

$$\{q\}_N = \left[\left\| \begin{array}{c} \\ \end{array} \right\| \right]^T_{N \times N_R} \left[\begin{array}{c} q_R \end{array} \right]$$

G. Vermot des Roches, « Frequency and time simulation of squeal instabilities. Application to the design of industrial automotive brakes », Ph.D. ECP, CIFRE SDTools, 2011
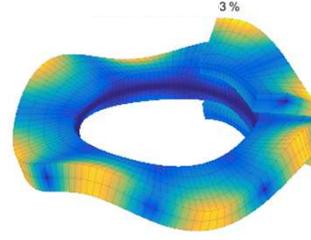
- Start $\mu = 0$ small damping
- increase μ : coupling and transition towards instability

Normal

Tangent
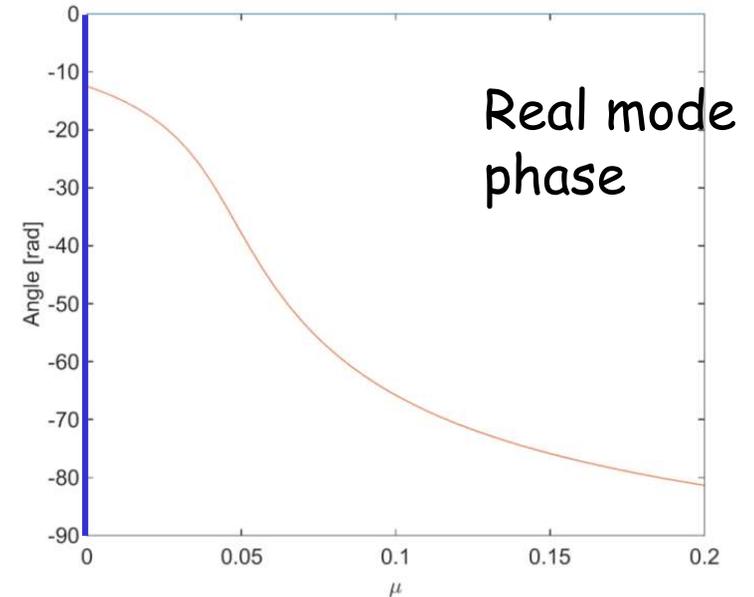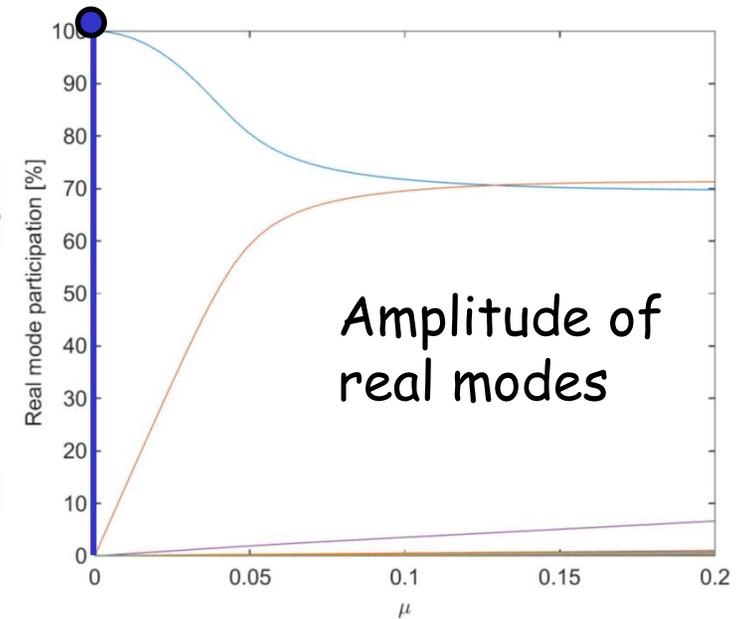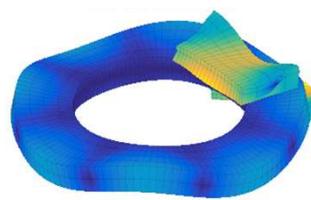
Normal

5047 Hz 0.03 %

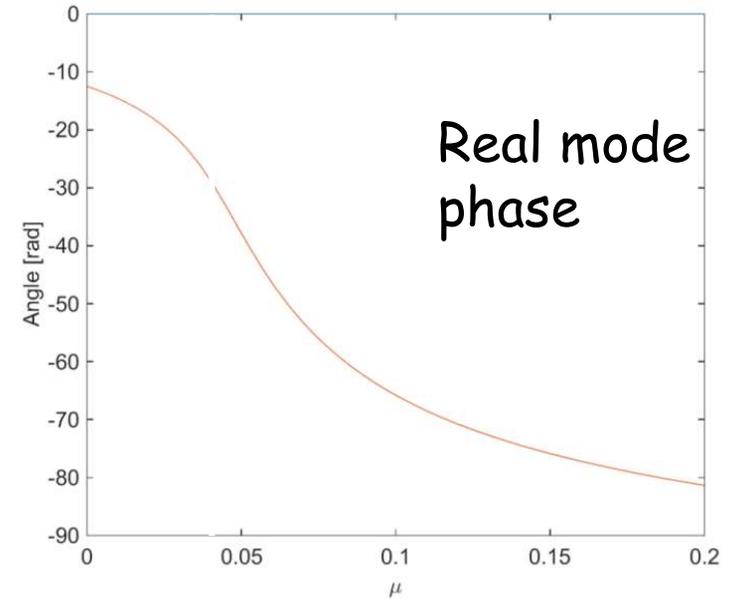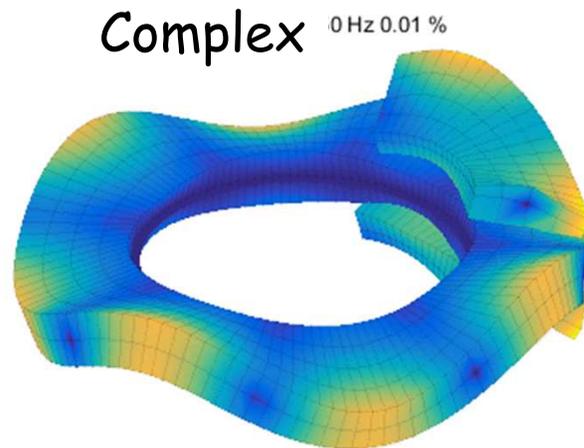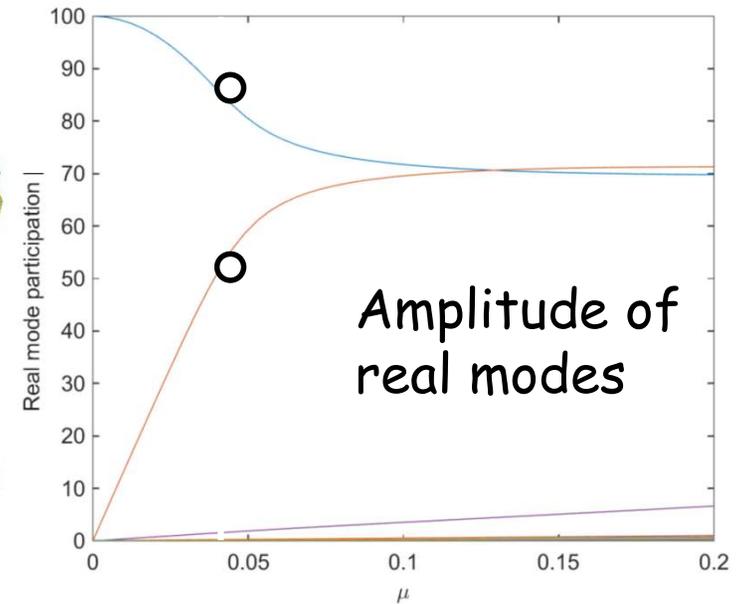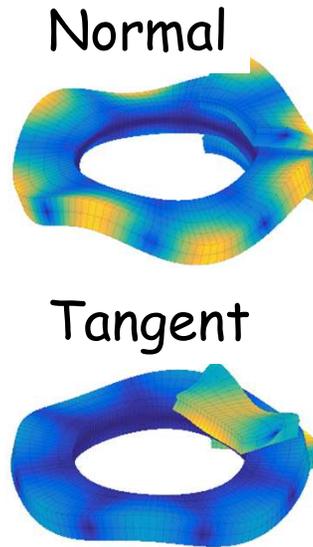Amplitude of real modes

Real mode phase
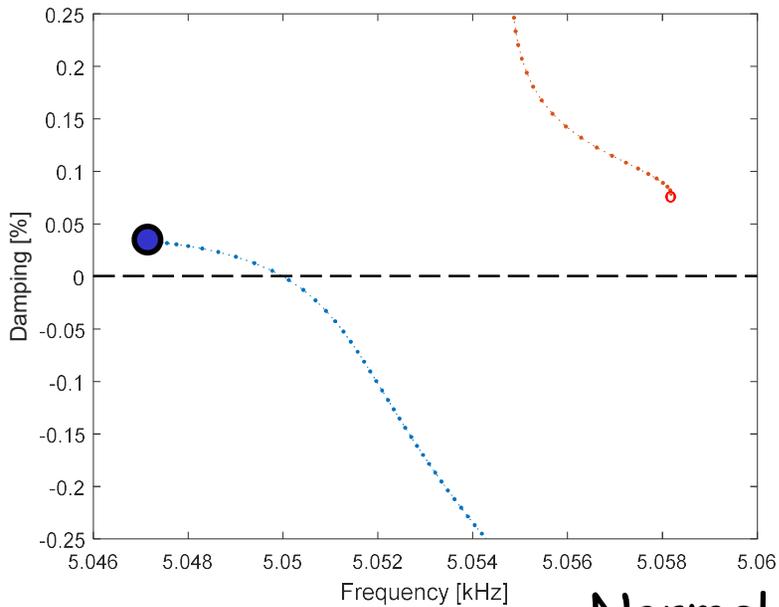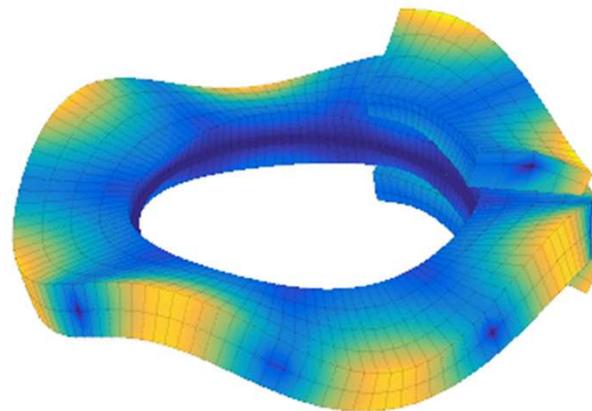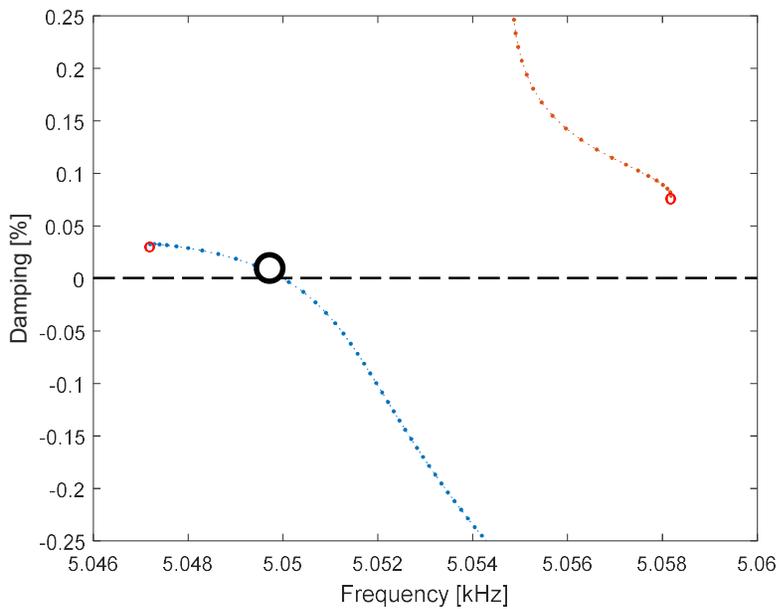
Participation of real shapes to complex modes

Normal

Tangent

Complex

Amplitude of real modes

Real mode phase

Subspace learning & basis selection

$$\left[\begin{array}{c} y(c, f_{\max}(t)) \quad \rightarrow \quad \textit{time} \\ \downarrow \\ \textit{sensors} \end{array} \right]$$

More details : G. Martin ISMA 2018

$y(f_{max}(t), t)$ **Time varying Complex shape** = **Space :** Principal real shapes **Time :** Complex amplitudes
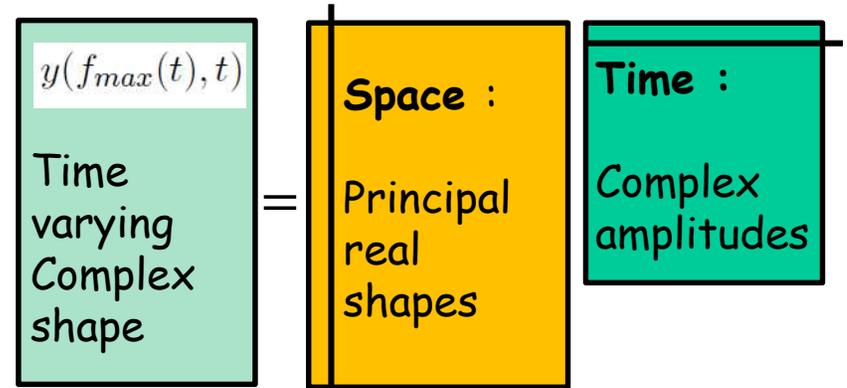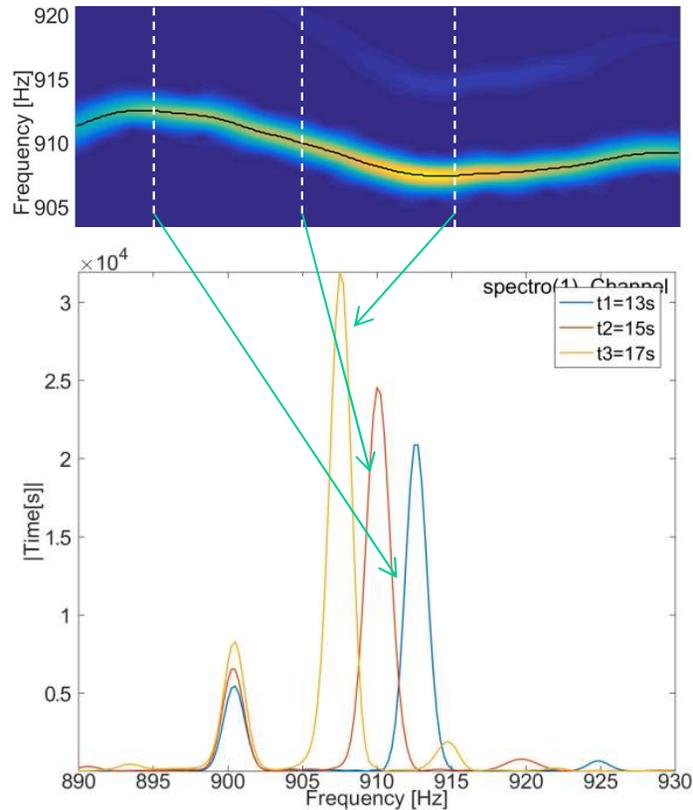
SVD $\sum_i \left\{ \begin{array}{c} U_i \\ \downarrow \\ sensors \end{array} \right\} \{ a_i(t) \quad \rightarrow \quad time \}$
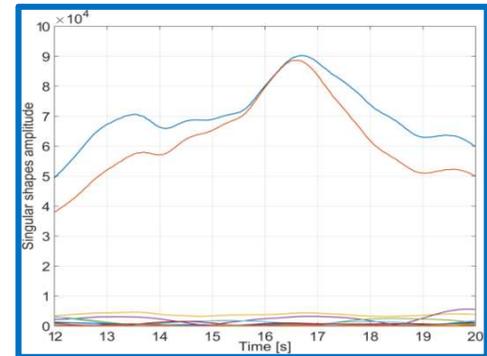
- only 2 significant real shapes & 2 associated DOF
- shapes independent of parameter
- Parameter : difficult to control (wheel position, brake event, …) but exists

LTI. Space frequency separation
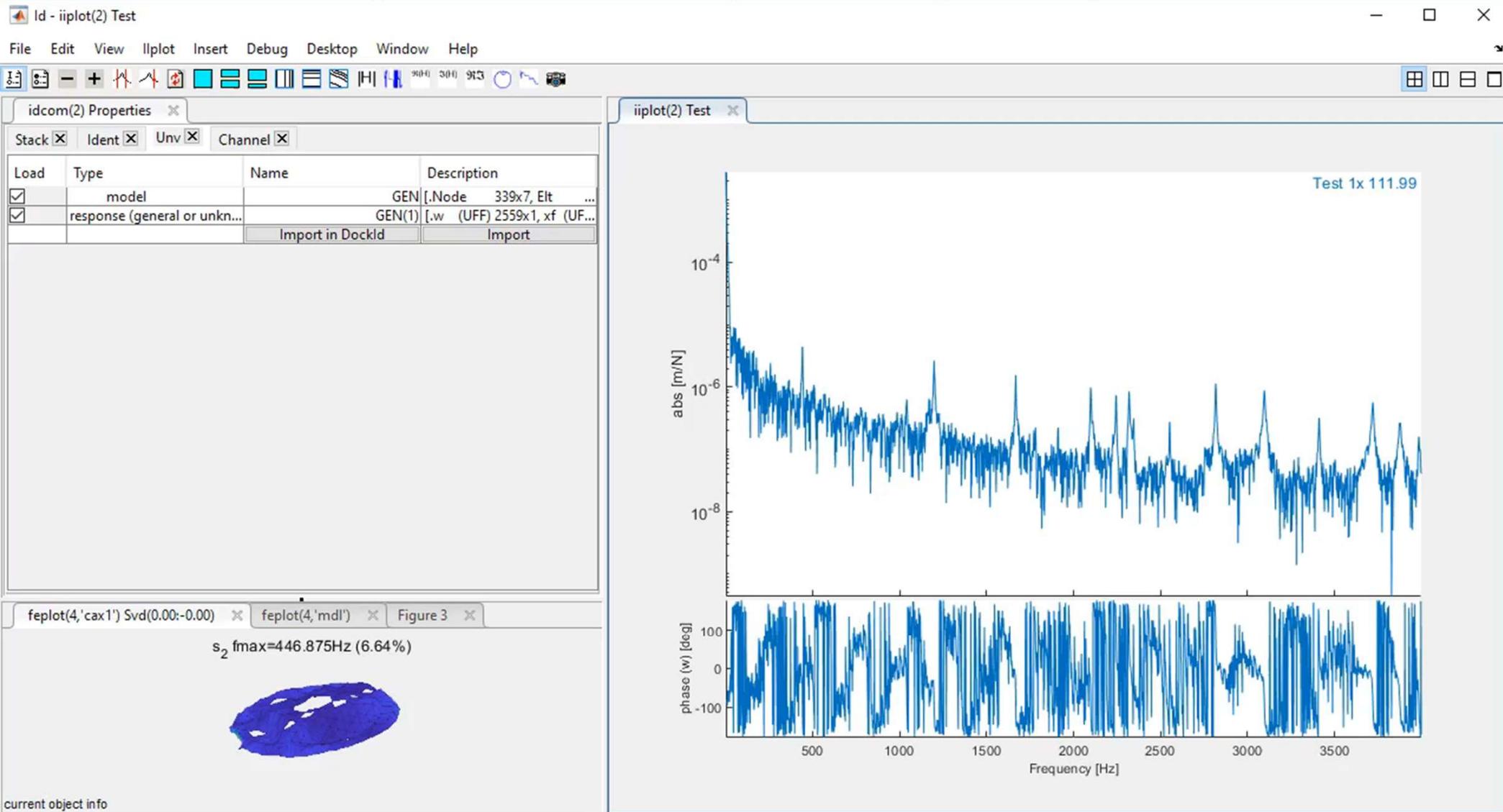
SVDCur button in SDT
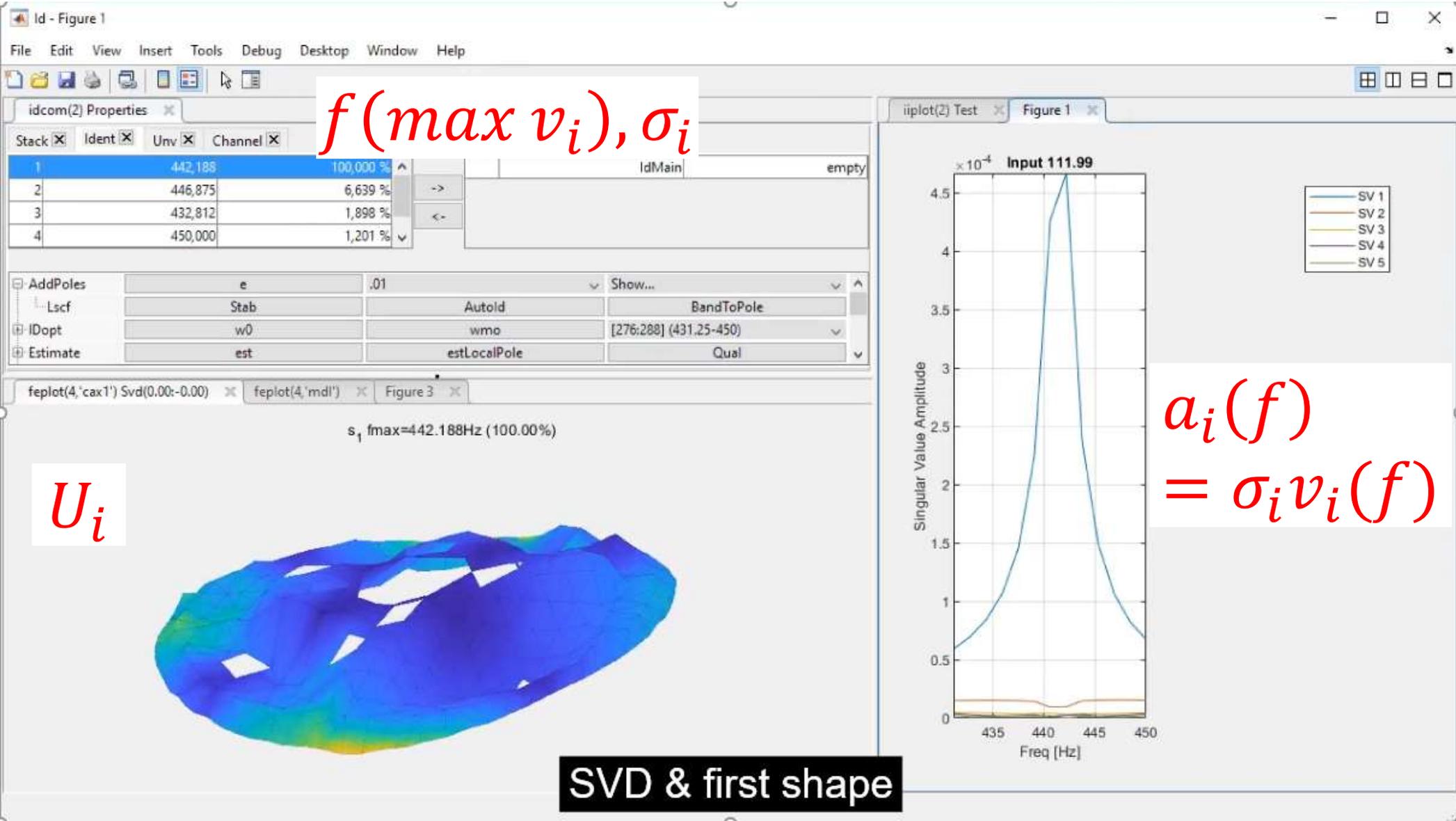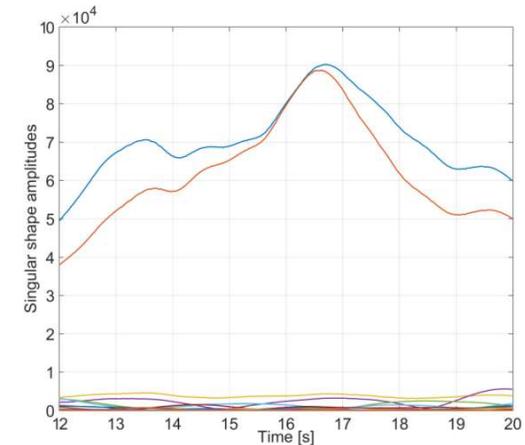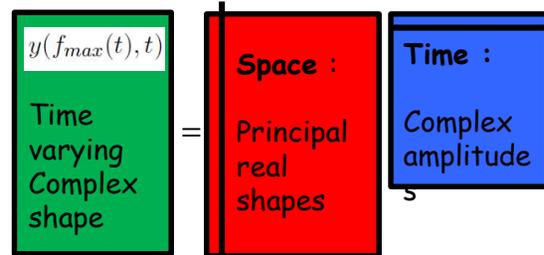
SVD & first shape

Outline

1. Squeal motivation

2. Subspace building strategies (learning phase) :
   - CMS : modes + static + parameters
   - Time & frequency snapshots, wave/cyclic computations, …
3. Physical vs. generalized/modal coordinates
4. Basis building strategies
   - Gram-Shmidt / LU : classical non-sorted
   - Sorting contributions : SVD
   - Choosing norms (SVD variants)
   - Analyzing right singular vectors (modal/generalized coordinates)

   - Other classic use : least squares & conditioning

Subspace generation

Traditional : modes + static correction

$$T = \begin{bmatrix} \phi(Zcc(\omega_j)) & K_{CC}(s)^{-1}K_{CV}(s)V_{In} \\ 0 & V_{In} \end{bmatrix}_{\perp M,K}$$

Snap-shot Ritz basis

$$T = \left[\left\{ \begin{array}{c} Z_{CC}(s)^{-1}Z_{CV}(s)V_{In} \\ V_{In} \end{array} \right\}_{s \in i\omega_{target}} \right]_{\perp M,K}$$

E055 Flex-BMI-P3

capacity PIEZO -60
capacity PIEZO -30
capacity PIEZO 0
capacity PIEZO 30
capacity PIEZO 60
capacity PIEZO 90
capacity PIEZO 120

Perm(T) p1 piezo ss

Et*0.50 Cap
Et*0.75 Cap
Et*1.00 Cap

Perm(T) p1 piezo FrfSnap

Et*0.50 Cap
Et*0.75 Cap
Et*1.00 Cap
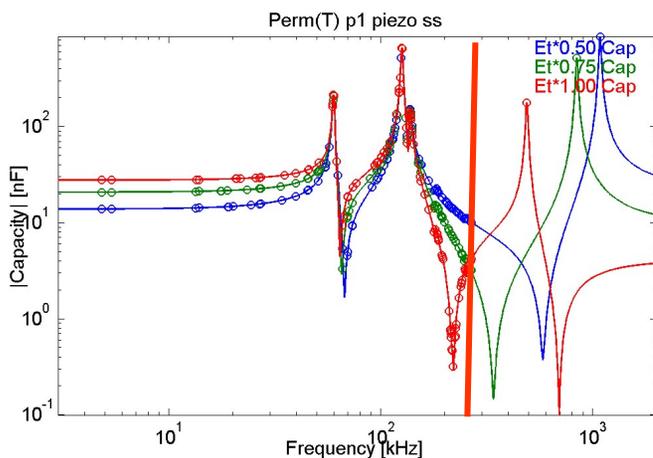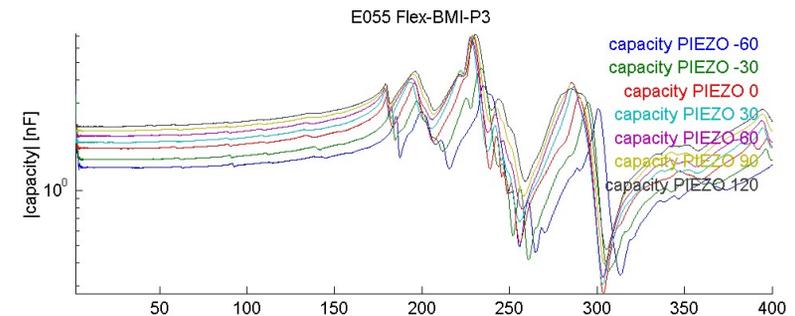
3 out of 100 useful modes
Relatively close static correction
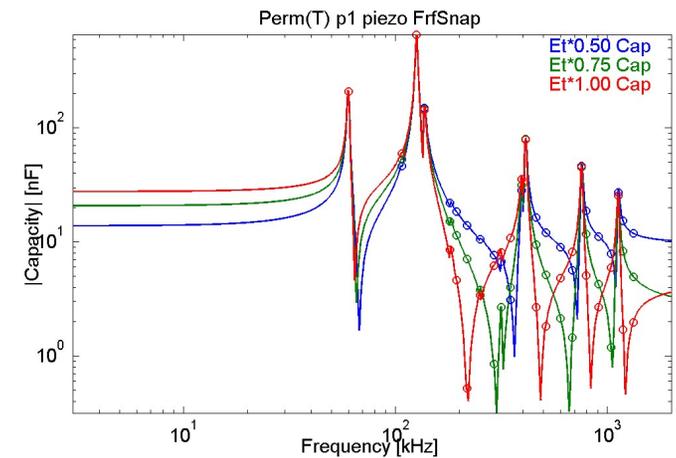
Easily captures wide range

Subspace & parametric model

- Parametric model reduction

    - Multi-model reduction : $T = [\phi(p_1)\ \phi(p_2)]_\perp$
    - First order correction : $T = [\phi(p_0)\ K^{-1}K_p\phi(p_0)]_\perp$

Environment
Design point

In → System → Out
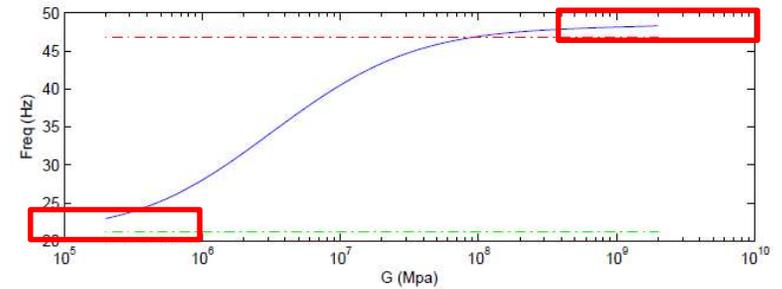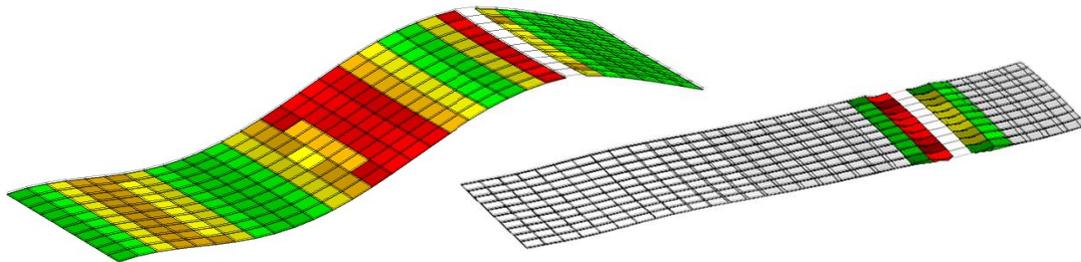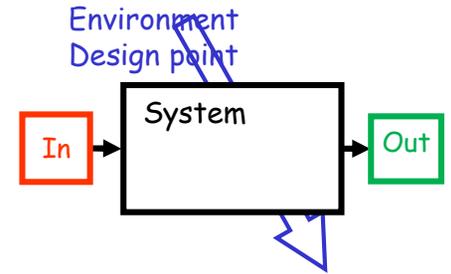
Interface reduction : wave/cyclic

Best interface reduction = learn from full system modes

1. Learn using wave (Floquet)/cyclic solutions
2. Build basis with left/right compatibility
3. Assemble reduced model
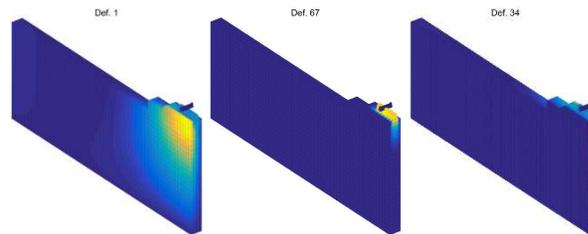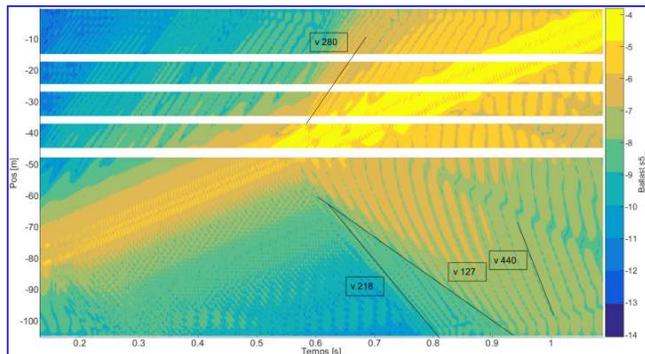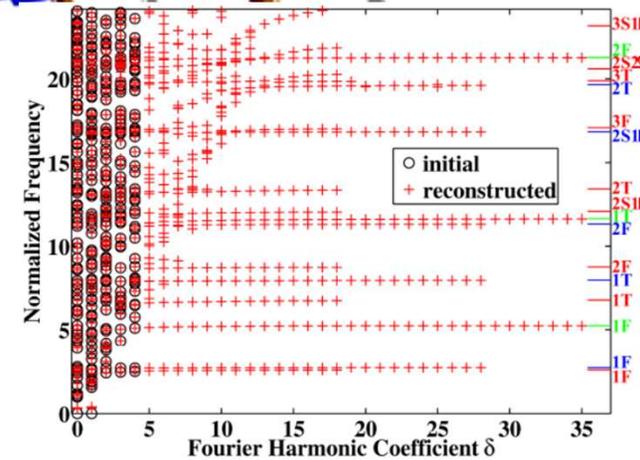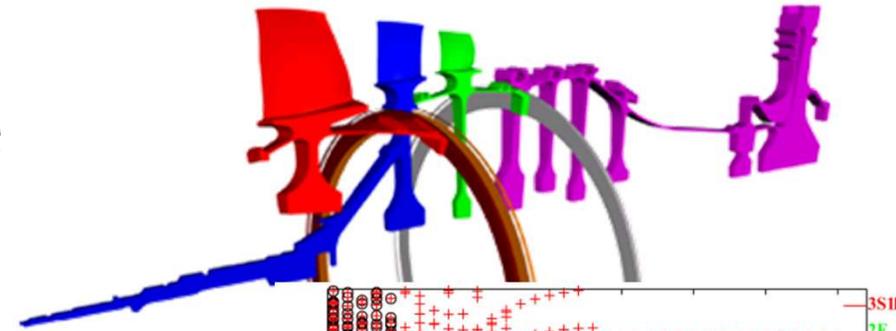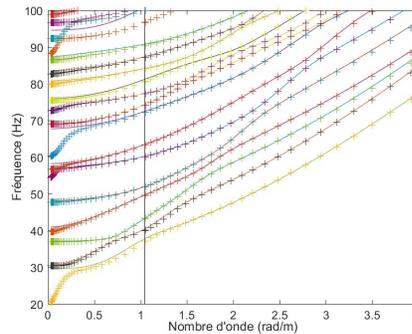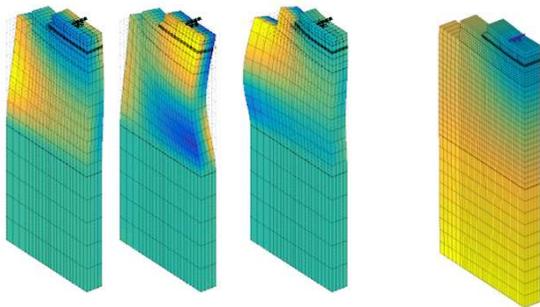
PhD Elodie Arlaud, 2016

PhD Sternshuss 2008

From shapes to bases

- Gram Schmidt (iterative)

- Iterative maximum sequence (LU)

- Missing notion of "optimal" order = SVD

https://www.code-aster.org/V2/doc/default/en/man_r/r5/r5.01.01.pdf

$A = U S V^H$

[A]{X}

{X}

space   time

$[r]_{\phi} = q_i(t_j)$ =

u_i

σ_i

T

v_i
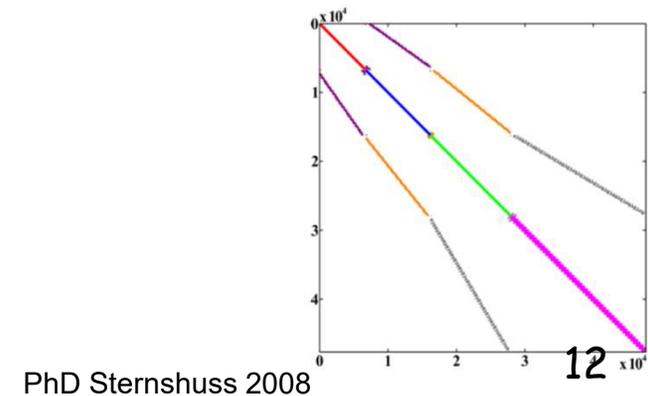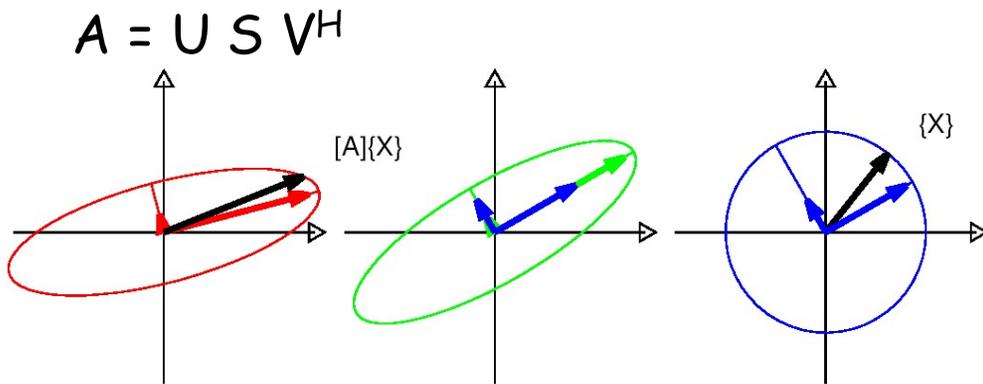
SVD

- {X} on sphere in input space transformed in {Y}=[A]{X} ellipsoid
- Sorted series of rank 1 contributions

shape   DOF
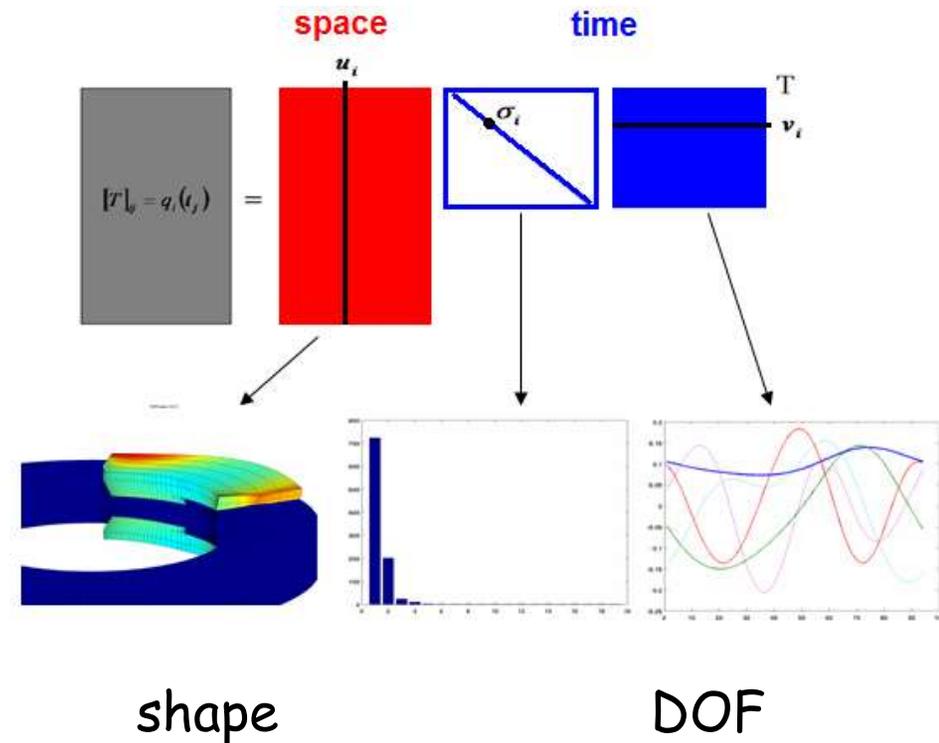
Video : MIT opencourseware Singular Value Decomposition (SVD)

Choosing norm : example modes

- {ϕ} on unit strain energy sphere output is kinetic energy

- Singular value $\frac{1}{\omega_j^2} = \frac{\phi_j^T M \phi_j}{\phi_j^T K \phi_j}$ = 1/Rayleigh quotient

AIAA Journal, Balmes, 1996

SVD, variants, related

Random fields Karhunen-Loeve [1] :
- input-norm I for all DOFs
- output norm spatial correlation
  $C = \exp[-(|x_1 - x_2| + |y_1 - y_2|)]$
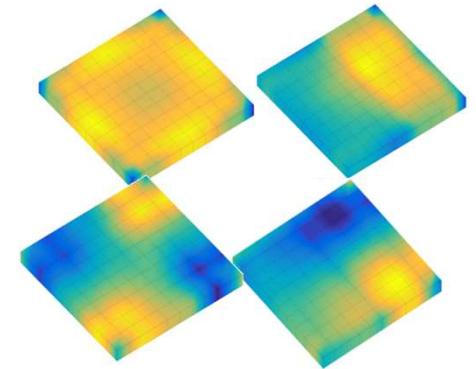
PCA Principal Component Analysis
POD based on snapshot-reduction [2] :
- input-norm I on snapshot vectors
- output norm I

Junction modes [3-4]
- input-norm I for modes
  or contact stiffness
- output norm local stiffness
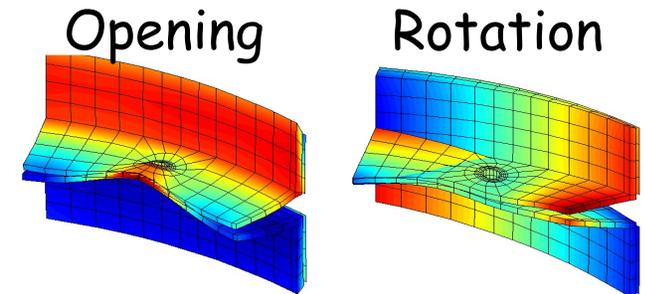
Non-linear dimensionality reduction (manifold) [5]
- More complex relation between parameters

[1] Chung, Gutiérrez, & all, "stochastic finite element models," IJME, 2005.
[2] Kershen & al. "POD", Nonlinear dynamics , 2005
[3] Balmes, Vermot, "Colloque assemblages 2015", + [4] Bendhia 1-epsilon compatibility EJCM 2010
[5] Ph.D. Olivier Vo Van 2016

Opening   Rotation
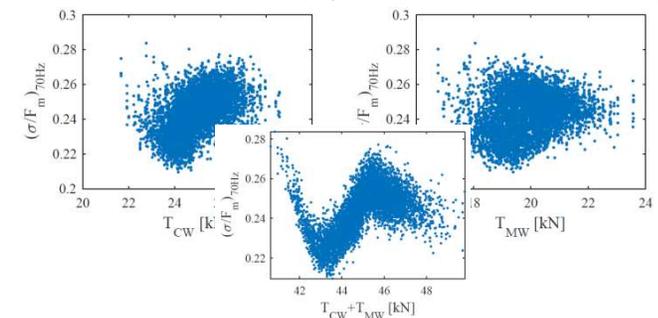
SVD uses : EMA multiplicity

MMIF (all channels of IIxf)
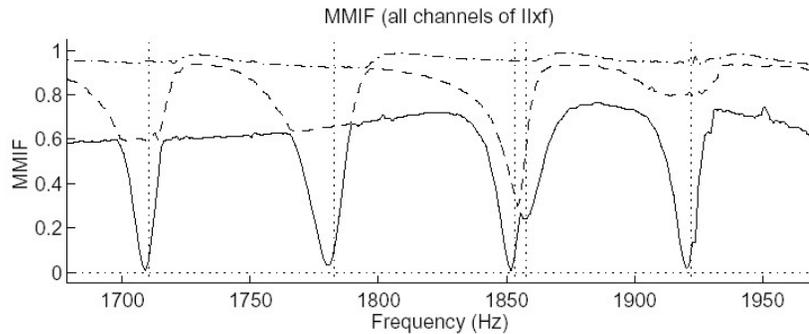
$$\frac{R_j}{s - \lambda_j} = \frac{\{c\psi_j\}\{\psi_j^T b\} + \{c\psi_{j+1}\}\{\psi_{j+1}^T b\}}{s - \lambda_j}$$

ω (Hz)	$\zeta\%$	LinLS	LogLS	σ_2/σ_1	σ_3/σ_1
1853.3	0.306	1.97e-8	1.22e+2	0.47	0.01
1853.3	0.306	1.07e-8	0.83e+2	0.03	0.01
1857.8	0.542			0.07	0.05

- Contributions of nearby poles can sometimes be grouped as a single multiple pole
- Identifying separate poles is better

Subspace m. : output only

$$R_i = E(Y_k Y_{k-i}^T)$$

$$[h(k\Delta t)]_{pq} = \begin{bmatrix} [R(0)] & [R(1)] & \cdots & [R(q-1)] \\ [R(1)] & [R(2)] & \cdots & [R(1+q-1)] \\ \vdots & \vdots & \ddots & \vdots \\ [R(p-1)] & [R(p-1+1)] & \cdots & [R(p-1+q-1)] \end{bmatrix}$$

$$G = E(X_k Y_k^T)$$

$$[h(k\Delta t)]_{pq} = [O_p(C,A)][C_q(A,G)]$$

$$[O_p(C,A)] = \begin{bmatrix} [C] \\ \vdots \\ [C][A]^{p-1} \end{bmatrix} \text{ and } [C_q(A,G)] = \begin{bmatrix} G & AG & \ldots & A^{q-1}G \end{bmatrix}$$

Subspace identification methods use SVD
See course notes section 7.5

Outline

1. Squeal motivation
2. Subspace building strategies (learning phase) :
   - CMS : modes + static + parameters
   - Time & frequency snapshots, wave/cyclic computations, …
3. Physical vs. generalized/modal coordinates
   - Using generalized coordinates makes engineering sense
   - Hyper-reduction of distributed NL is then possible

4. Basis building strategies
   - Gram-Shmidt / LU : classical non-sorted
   - Sorting contributions : SVD
   - Choosing norms (SVD variants)
   - Analyzing right singular vectors (modal/generalized coordinates)

   - Other classic use : least squares & conditioning

Physical & Modal DOF

- **Physical domain:**

$$[M]\{\ddot{q}(t)\} + [C]\{\dot{q}(t)\} + [K]\{q(t)\} = \{f(q, \dot{q}, t)\}$$
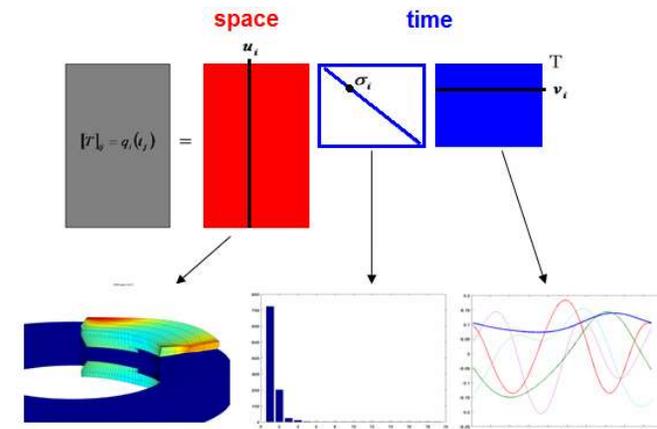

- **Modal domain:**

  - mass orthogonality condition $\phi^T M \phi = I$
  - stiffness orthogonality condition $\phi_j^T K \phi_j = \omega_j^2$
  - Modal equation

$$[I]\{\ddot{\alpha}(t)\} + [\Gamma]\{\dot{\alpha}(t)\} + \left[\diagdown \omega_j^2 \diagdown \right] \{\alpha(t)\} = \{f(\alpha, \dot{\alpha}, t)\}$$

- Modal amplitudes $\{\alpha\} = [\boldsymbol{\phi^{-1}}]\{q\} = [\boldsymbol{\phi^T M}]\{q\}$
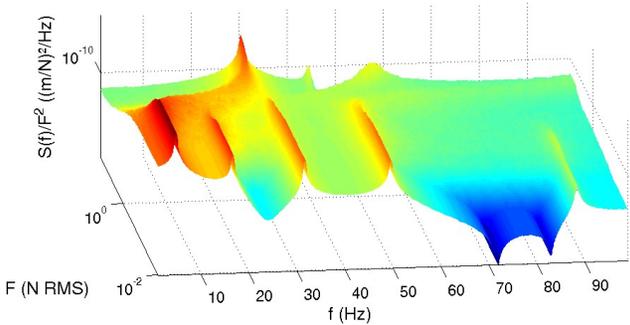
   Associated concepts : force appropriation, modal filter
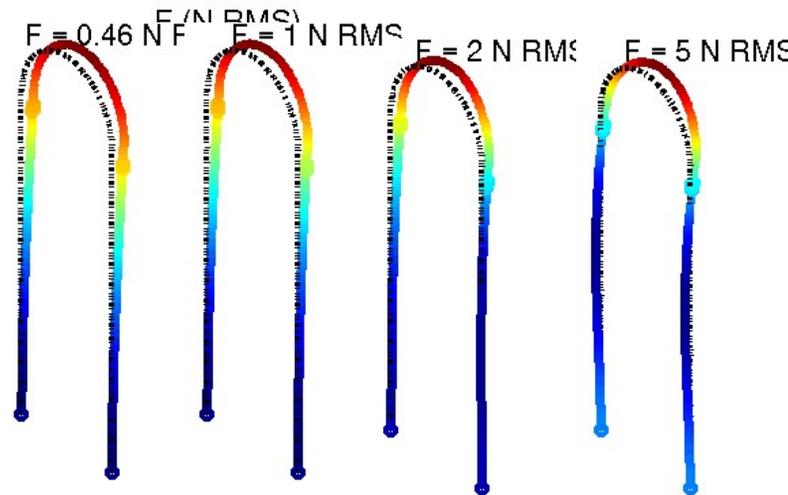
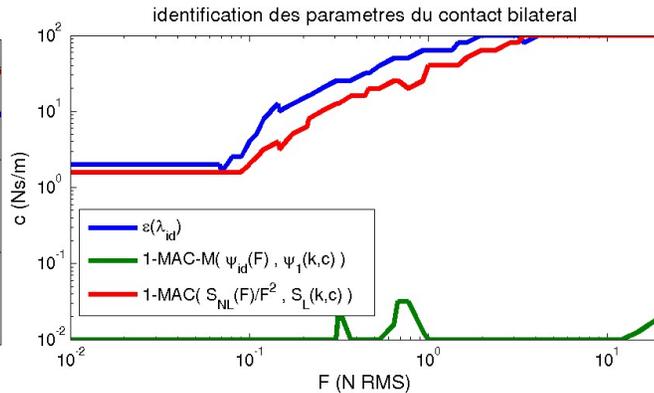- Modal energies $e_j = \frac{1}{2}\left(\dot{\alpha}_j^2 + \omega_j^2 \alpha_j^2\right)$

Modal participations in ODS

depl_210.01 (m)
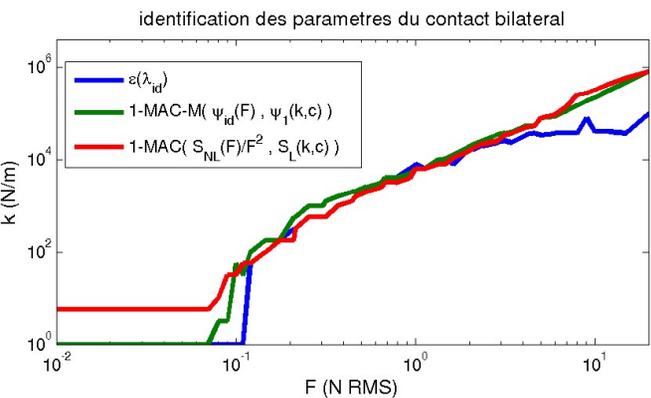

participation a plus de 10 % de l'ODS

- Extract shape = SVD around « resonance »

- Obtain modal amplitudes of nominal modes

- Apparent stiffness/damping consistent for various methods

F = 0.46 N F   F = 1 N RMS   F = 2 N RMS   F = 5 N RMS

identification des parametres du contact bilateral

identification des parametres du contact bilateral

PhD. Thénint 2011
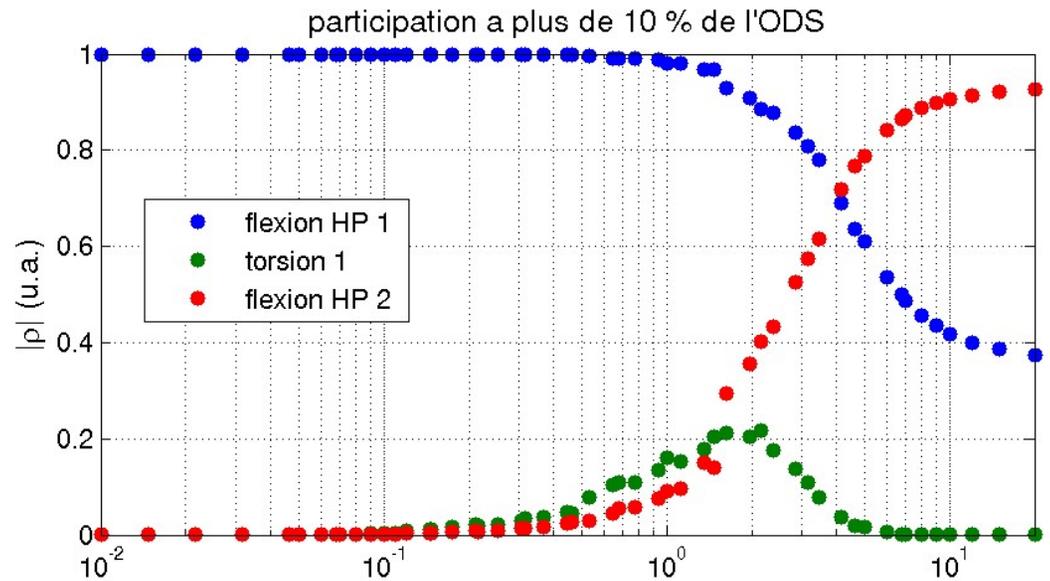
Tracking modes

- Horizontal lines : global modes
- S curve : local cable guide mode

Sensors can help
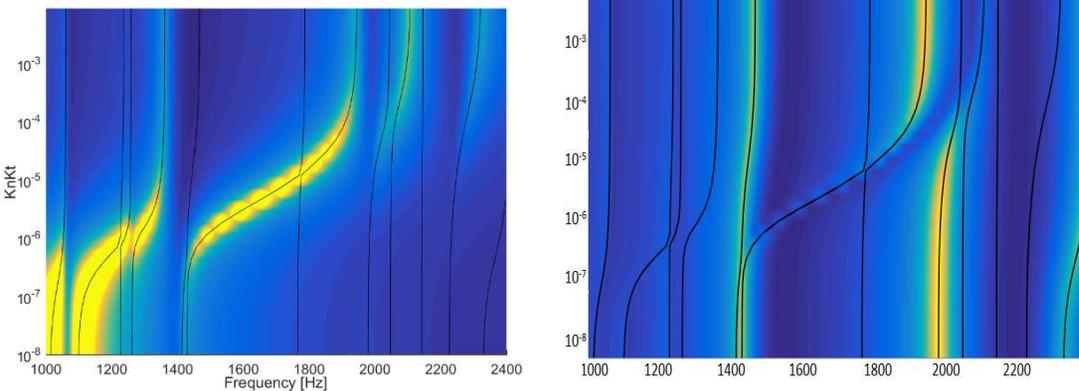
- But can we track modes better ?

Assembly

Rigid plate

coef=1.91e-05
1873.58 Hz
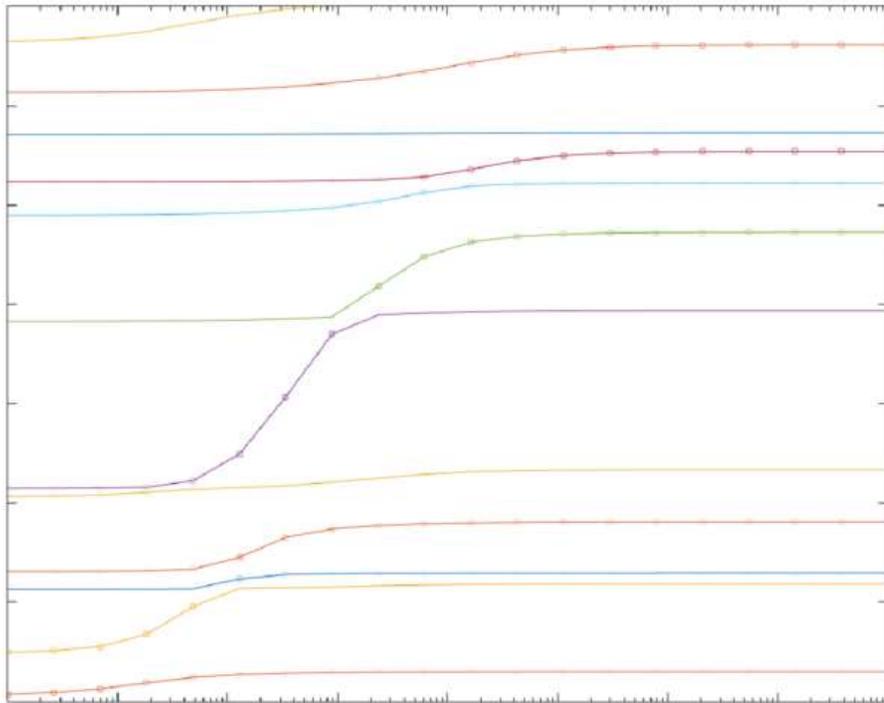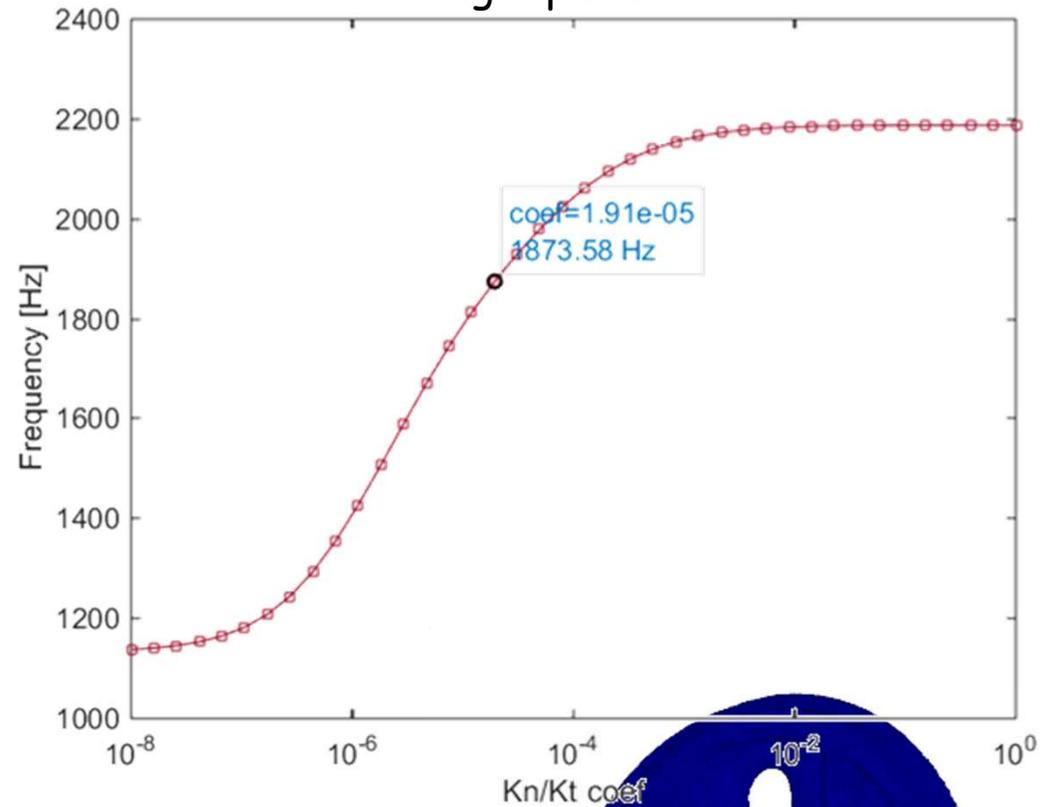
$$\{q\} = \begin{bmatrix} RB_1 & & 0 \\ & \ddots & \\ 0 & & I \\ & & & \ddots \end{bmatrix} \{q'\}$$

Plate : rigid motion

Cable guide all DOFs

Modal energy computations
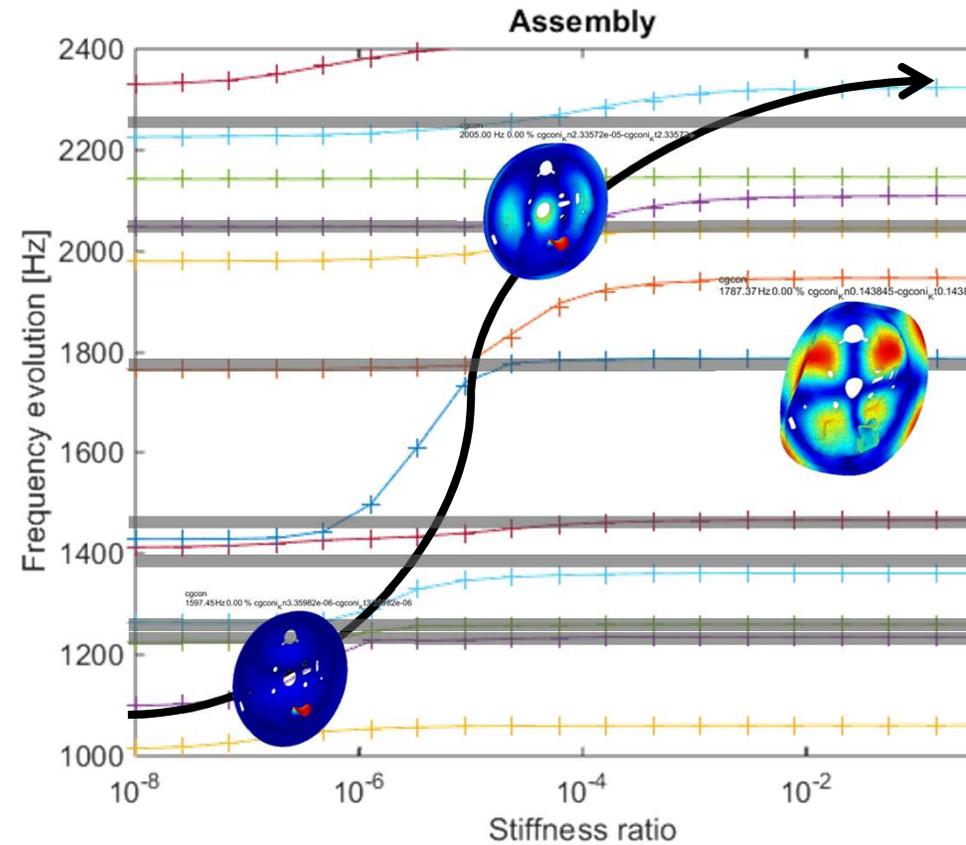
- Does the shape change in NL behavior

Mode 1 at 41.61 Hz

$$2E_{mj}(t) = \omega_j^2 \alpha_j^2(t) + \dot{\alpha}_j^2(t)$$

$E_p(t)$ $E_k(t)$

Vermot & Balmes, ISMA 2016   23
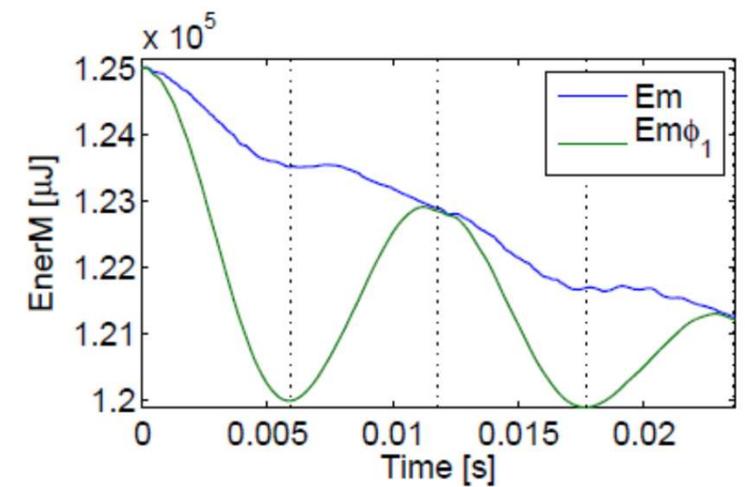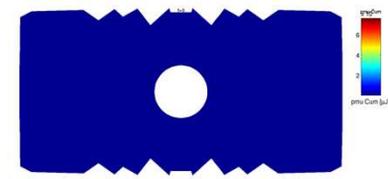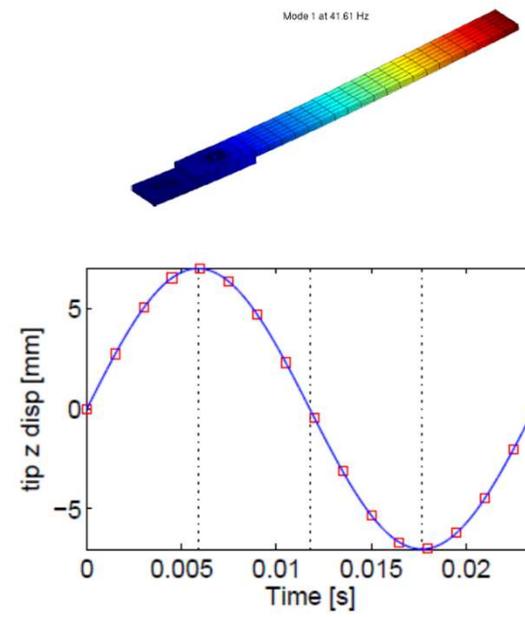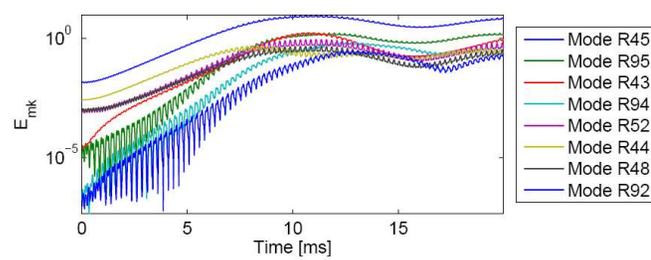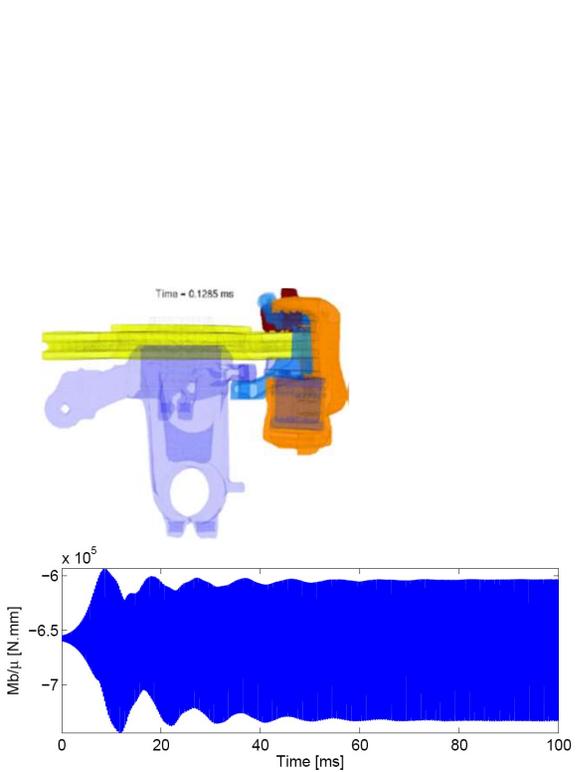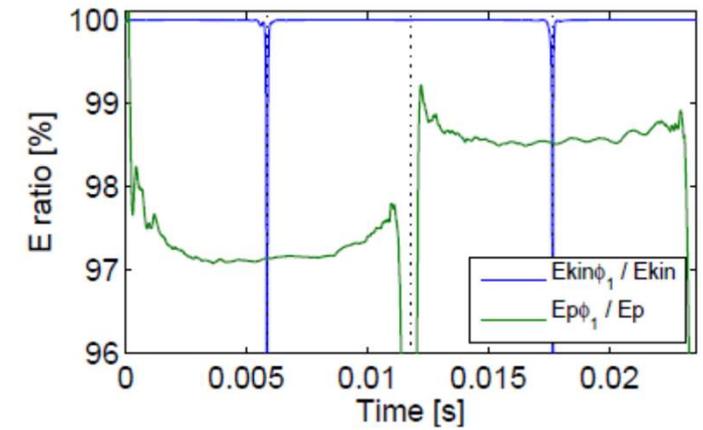
Hyper reduction of transient NL

HR : generic form of FEM problems

1. FEM kinematics $u(x,t) = [N(x)]\{q(t)\}$

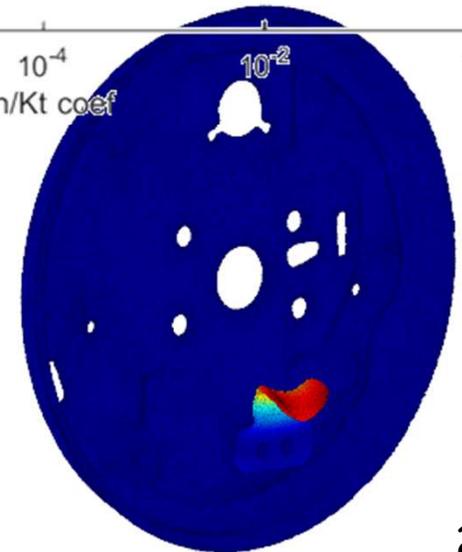2. Material evolution : $S = f_{mat}(\nabla u, u_{int})$

3. Virtual work $\quad M\ddot{q} + \mathcal{F}_{int}(q, u_{int}) = \mathcal{F}_{ext}$

- Integration at quadrature points g (disassembly)

$$\mathcal{F}_{int}(q, u_{int}) = \sum_{g} \mathbb{C}^T J_g w_g f_{mat}(\epsilon, u_{int})$$

$$= [\mathbb{B}]_{N \times (N_g \times N_f)} \left\{ f_{mat}\left([\mathbb{C}]_{(N\epsilon \times N_g) \times N}\{q\}, u_{int}\right)\right\}$$
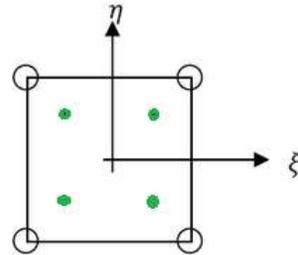
HR phase 2 : kinematic reduction

1/ Off-line learning (non-linear High Fidelity Simulation)
   Snapshots or iterative method

2/ Kinematic reduction using SVD, or CMS

$$q_{learn} = \sum_i \{U_i(x)\}\,(\sigma_i\,v_i(t))$$
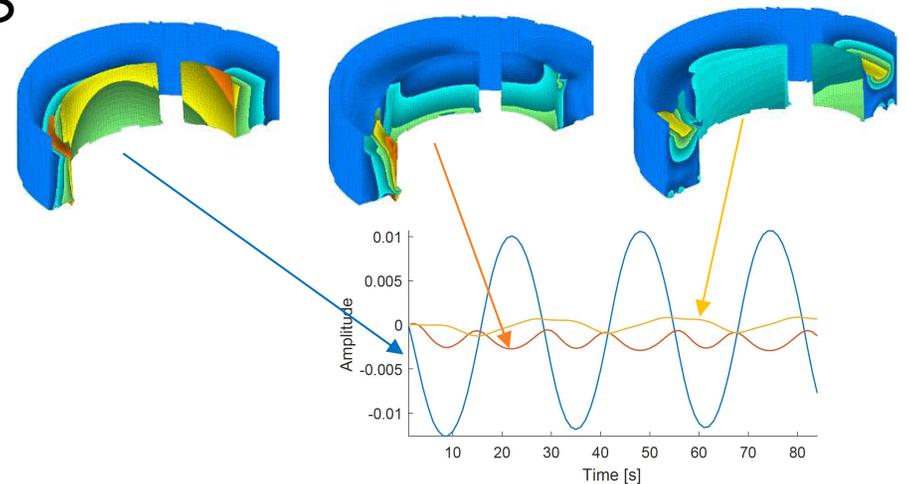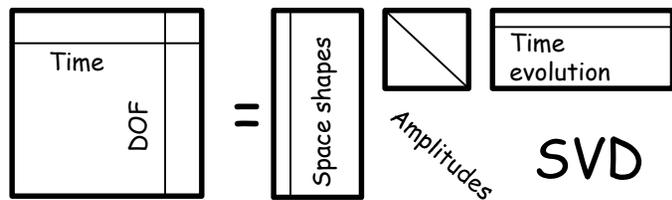
Reduced equations of motion $\quad T^T M T \ddot{q}_r + [T^T \mathbb{B}]_{Nqr \times Ng}\, f_{mat}(\mathbb{C}Tq_r, u_{int}) = T^T F_{ext}$

- N_g (gauss points) remains large
- $f_{mat}(\mathbb{C}Tq_r, u_{int})$ evaluation dominates

3/ Operator reduction (hyper-red.) = less points + new weights

$$\{F_{int}\} = \int_\Omega f_{mat}(x_g, t)\, \mathrm{d}V \approx \sum_g \mathbb{C}^T J_g w_g f_{mat\,g} = \mathbb{B}(x_g) f_{mat\,g}(q, t)$$

Choose NT learning points and minimize $\|w_g^*\|_0$ subject to[1]

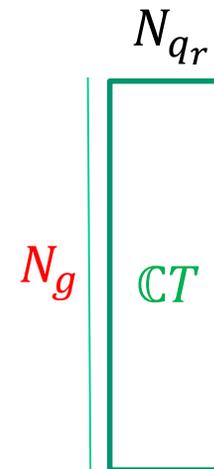$$\left\| \left[[T^T \mathbb{B}]\, S_l \right]_{(NT \times NR)} - \left[[T^T \mathbb{B}_g]\, S_{lg} \right]_{(NT \times NR) \times Ng} \{w_g^*\} \right\|_2 < \varepsilon_{tol} \text{ and } w_g^* > 0$$

$$T^T \mathbb{B}$$
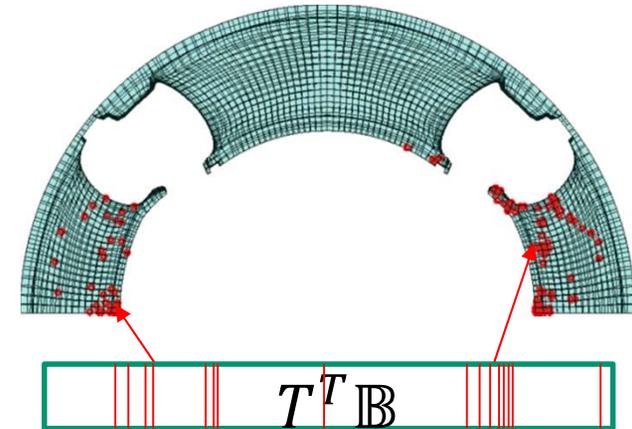
4/ On-line usage (29s vs. 27h)

5/ Post-processing (estimate continuous fields)
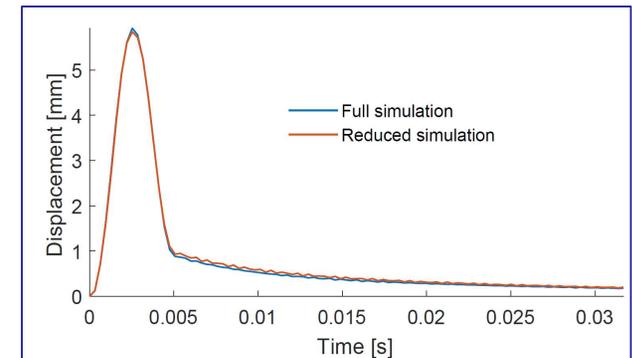
[1] Farhat, Avery, Chapman, Cortial, « Dimensional reduction of nonlinear finite element dynamic models with finite rotations and energy-based mesh sampling and weighting for computational efficiency», *IJNME*, vol. 98, nᵒ 9, p. 625-662, 2014.

[2] F. Casenave, N. Akkari, F. Bordeu, C. Rey, and D. Ryckelynck, "A nonintrusive distributed reduced-order modeling framework for nonlinear structural mechanics—Application to elastoviscoplastic computations," IJNME, vol. 121, no. 1, pp. 32–53, Jan. 2020, doi: 10.1002/nme.6187

[3] Penas, Rafael, *Models of dissipative bushing in multibody dynamics*, PhD ENSAM 2021

System = macro / micro

- Field distribution : macroscopic law not a simple function of material behavior

Micro
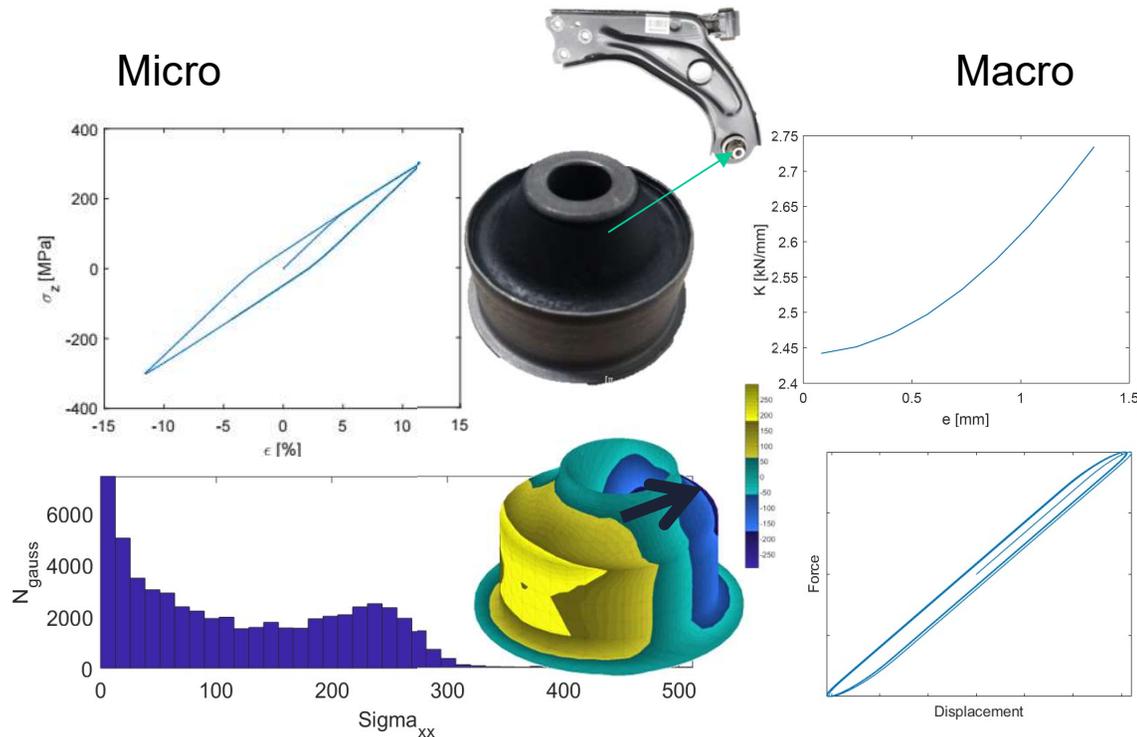
Macro

- Resultant (average in space)
- Loss factor (average in time & space)

$$\eta(X, T) = \frac{\frac{1}{2\pi} \int_0^T \sigma(X,t)\dot{\epsilon}(X,t)dt}{\frac{1}{2} \max_0^T (\sigma(X,t)\epsilon(X,t))}$$

System/macro
force-state map
F = f(q,t)
Meta-model
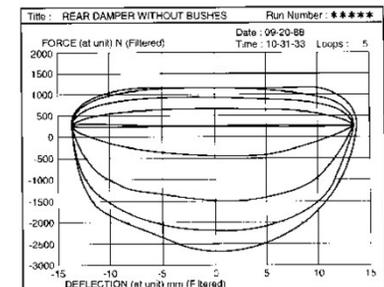
Internship/thesis topic 2024 @ SNCF

- Space-time scales & contact

- Impulse response representations

- Hyper-visco-hysteretic pads
- Pad hyper reduction

- Design studies

From : L. Pesaresi, JSV 2018

Forces (Fz target 117.72) Fz

etienne.balmes@ensam.eu , olivier.vovan@sncf.fr
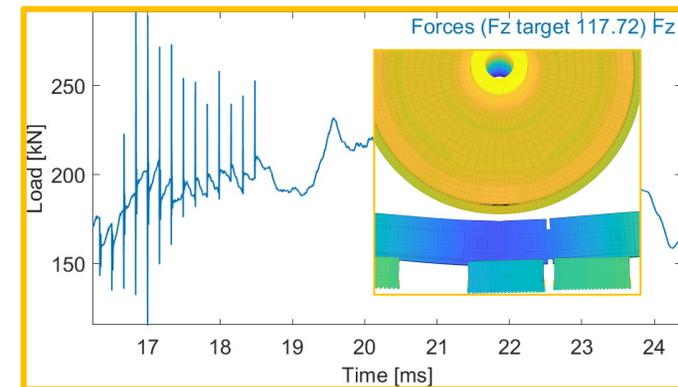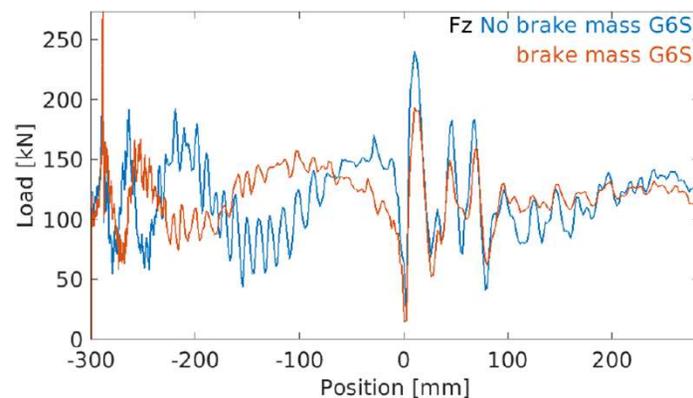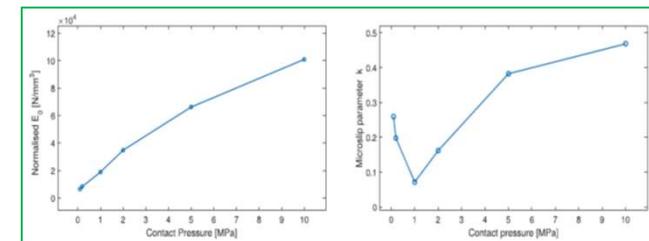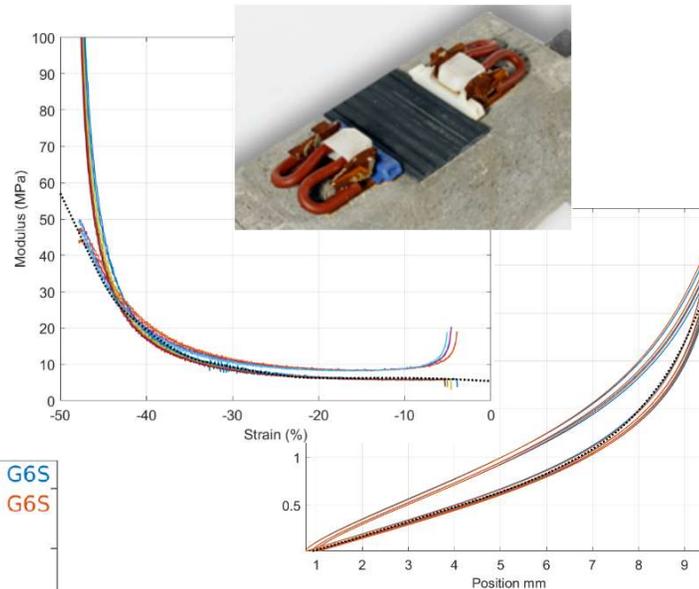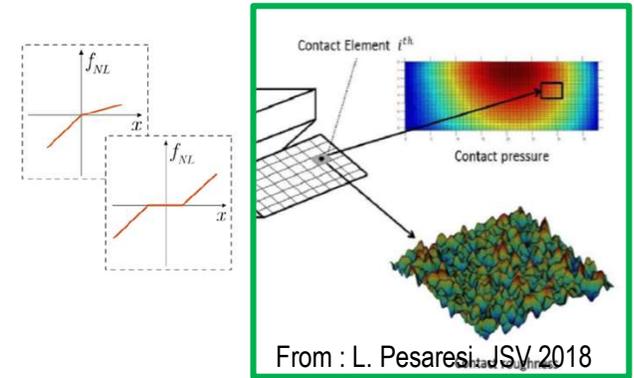
- $dt = 10^{-7}s$, speed $80\ m/s$, $8\ \mu m/step$
- Space scale: $300\ \mu m/gauss \Leftrightarrow$ time scale $3.75\ \mu s = 37.5\ steps$
- Wave front 1-10 MHz

Outline

1. Squeal motivation : Even in fairly complex case « the response occurs with a restricted subspace » = shapes remain invariant in some sense
2. Subspace building strategies :
   - CMS : modes + static + parameters
   - Time & frequency snapshots, wave/cyclic computations, …
3. Basis building strategies
   - Gram-Shmidt / LU : classical non-sorted
   - Sorting contributions : SVD
   - Choosing norms (SVD variants)
   - Analyzing right singular vectors (modal/generalized coordinates)

   - Other classic uses of SVD : least squares & conditioning

29
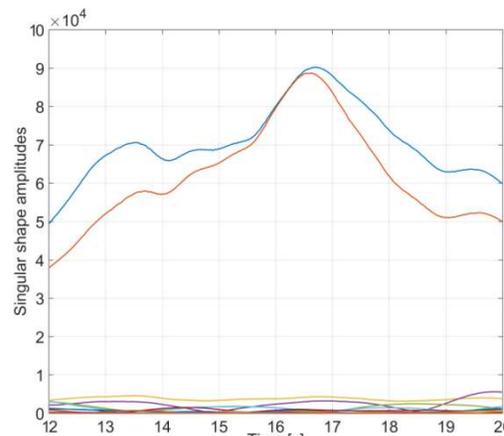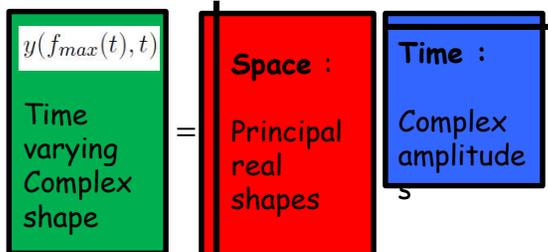
Matrix form $A = U\,S\,V^H$

Series form
$$[A] = \sum_{j=1}^{\min(n,m)} \sigma_j \{U_j\}\{V_j\}^T$$

Standard properties

$$\sigma_{\max}(A^{-1}) = \frac{1}{\sigma_{\min}(A)}$$

$$\sigma_{\min}(A^{-1}) = \frac{1}{\sigma_{\max}(A)}$$

$$\sigma_{\max}(A) - 1 \le \sigma_{\max}(I + A) \le \sigma_{\max}(A) + 1$$

$$\sigma_{\min}(A) - 1 \le \sigma_{\min}(I + A) \le \sigma_{\min}(A) + 1$$

$$\sigma_{\max}(A + B) \le \sigma_{\max}(A) + \sigma_{\max}(B)$$

$$\sigma_{\max}(AB) \le \sigma_{\max}(A)\sigma_{\max}(B)$$

Least squares and SVD

- SVD of A is of the form

$$[A] = \sum_{j=1}^{\min(n,m)} \sigma_j \{U_j\} \{V_j\}^T$$

- Least squares solution given by

- This is a Moore-Penrose pseudo-inverse

$$\{x\} = \sum_{j=1}^{\min(n,m)} \sigma_j^{-1} \{V_j\} \{U_j\}^T [b] = [V] \left[\backslash \sigma_j^{-1} \backslash\right] [U] [b]$$

$$\left[A^+\right] [A] \left[A^+\right] = \left[A^+\right]$$

$$[A] \left[A^+\right] [A] = [A]$$

$$\left([A] \left[A^+\right]\right)^T = \left([A] \left[A^+\right]\right)$$

$$\left(\left[A^+\right] [A]\right)^T = \left(\left[A^+\right] [A]\right)$$

Least squares conditioning

- One solves LS $\quad \min_{\{x\}} \| [A]\ \{x\} - [b] \|_2^2$

- But errors $\quad ([A] + [\delta A])\ (\{x\} + \{\delta x\}) \ =\ (\{b\} + \{\delta b\})$

- Problem is well conditioned if :

$$\frac{\|\delta A\|}{\|A\|} \ll 1, \qquad \frac{\|\delta b\|}{\|b\|} \ll 1 \qquad \Longrightarrow \qquad \frac{\|\delta x\|}{\|x\|} \ll 1$$

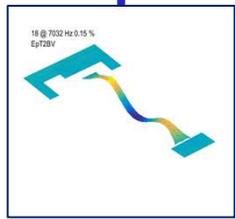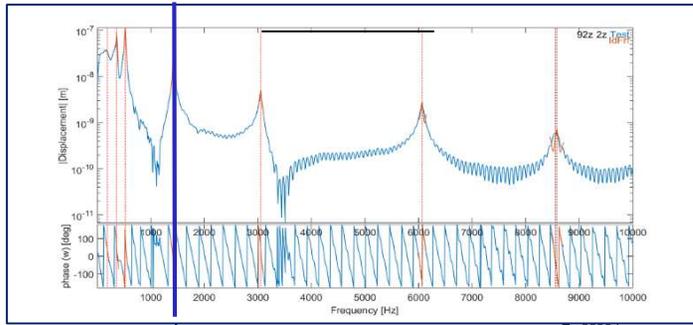- One can prove that (k condition number)

$$\frac{\|\delta x\|}{\|x\|} \ \leq\ \kappa(A) \left(\frac{\||\delta A\||}{\||A\||} + \frac{\|\delta b\|}{\|b\|} \right) \qquad\qquad \kappa(A) \ =\ \||A\|| \ \||A^{-1}\||$$

Non linear least squares

$$J(p) = \|R\|_2^2 = \mathrm{Trace}\left(R^T R\right) = \sum_{i,j} \bar{R}_{ij} R_{ij}$$

- First derivative
$$\frac{\partial J(p)}{\partial p} = 2\mathrm{Trace}\left(\frac{\partial R^T}{\partial p} R\right)$$

- Second derivative
$$\frac{\partial^2 J(p)}{\partial p^2} = 2\mathrm{Trace}\left(\frac{\partial R^T}{\partial p}\frac{\partial R}{\partial p} + R^T \frac{\partial^2 R}{\partial p^2}\right)$$

- Newton method $p^{n+1} = p^n + \delta p^{n+1}$ with
$$\left[\frac{\partial^2 J(p)}{\partial p^2}\right]\{\delta p^{n+1}\} + \left\{\frac{\partial J(p)}{\partial p}\right\} = \{0\}$$

- Convergence
$$\lambda_{\max}\left(\left[\frac{d^2 J}{dp^2}(p)\right]\right) < 1$$

Parameter equivalence : material example

Vertical bending

$$E_1 = argmin(\left| f_{flexion}(E_1, G_{31}) - f_{flexi\ ,obj} \right|)$$
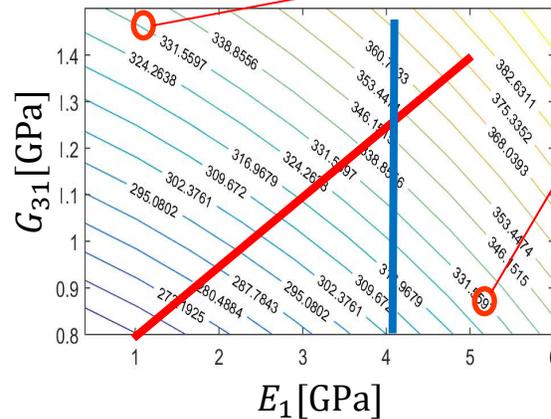
E_1
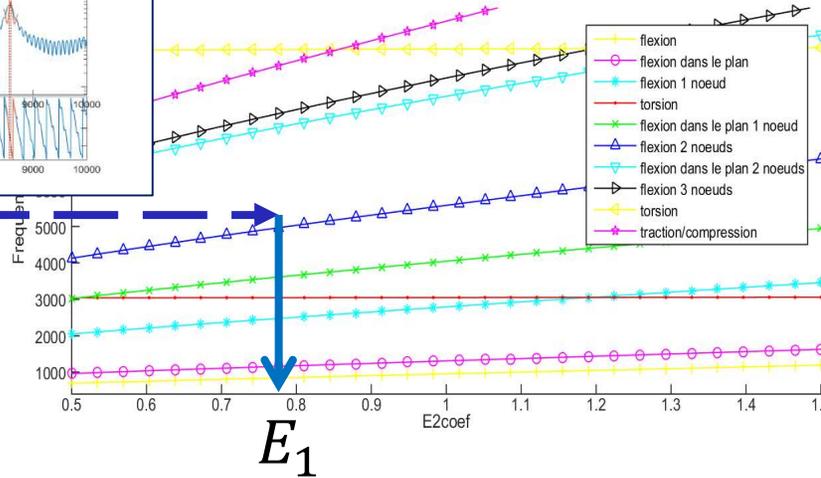
Assumption must be made

- G_{31}/E_1 constant ?
- E_1 traction ?

Solution is not unique
Mode-shape very similar
Jacobian poorly conditionned

SVD and conditioning

- For $||\delta A||=\varepsilon$ smaller eigenvalues need to be neglected (because not well known)

$$[\Sigma] = \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_{NR} & & & \\ & & & \epsilon & & \\ & & & & \ddots & \\ & & & & & \epsilon \end{bmatrix}$$

- Hence well conditioned pseudo-inverse

$$\{x\} = [A]^+ [b] = \left[\sum_{j=1}^{NR} \sigma_j^{-1} \{V_j\} \{U_j\}^T \right] [b]$$
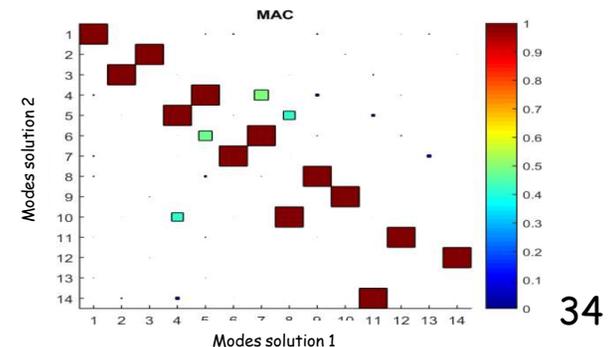
Regularization methods

- One penalizes parameter changes

$$\min_{\{\delta p\}} \; J(p^n) \; + \; \sigma \; \{p^n - p^0\}^T \{p^n - p^0\}$$

- Iterations of the form

$$\left[\frac{\partial^2 J(p)}{\partial p^2} + \sigma \, [I]\right] \{\delta p^{n+1}\} + \left\{\frac{\partial J(p)}{\partial p} + \sigma \, \{p^n - p^0\}\right\} = \{0\}$$

- Small singular values replaced by σ
- Large singular values not modified

$\Rightarrow$ Similar effect than pseudo-inverse
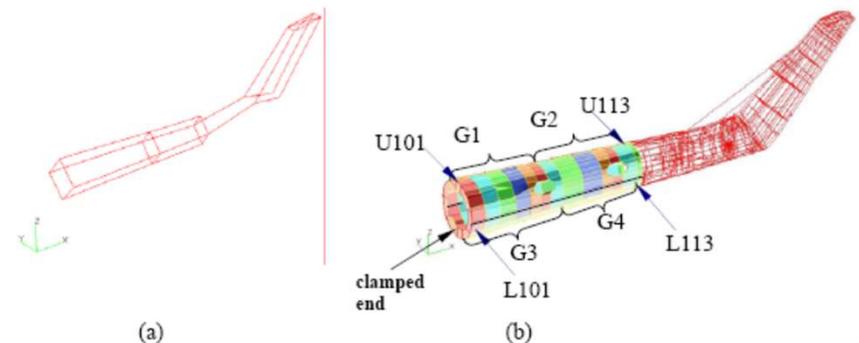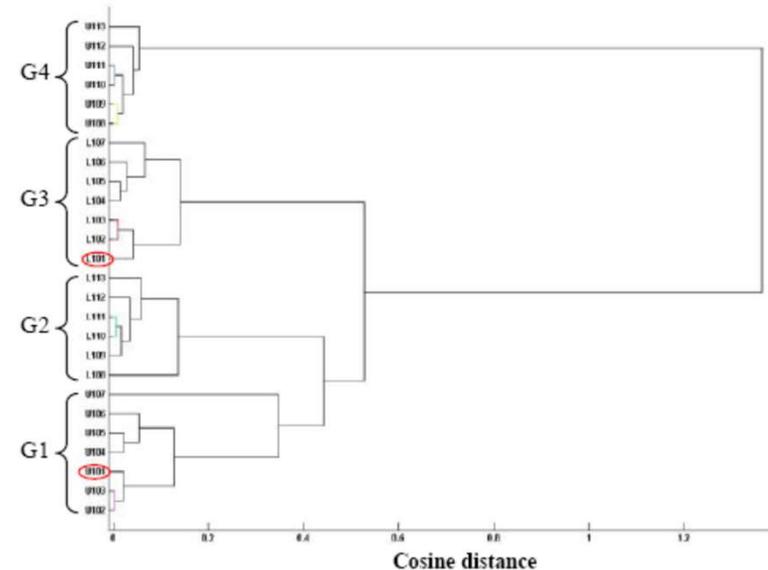   with truncation of small singular values

Sample clustering method

- Compute cosine distance

$$\cos^{-1}\left(max \frac{\langle u|v\rangle}{\|u\|\|v\|}\right)$$

$$A,B \rightarrow U_A, U_B \rightarrow \sigma_{max}\left(U_B - U_A(U_A^H U_B)\right)$$

- Recursively group elements with smallest distance
- K-means algorithm

Clustering of Parameter Sensitivities: Examples from a Helicopter Airframe Model Updating Exercise. Shahverdi, Mottershead & All