

ENSAM / PIMM
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SDTools

Methods for vibration design and validation
Méthodes de conception et de validation en vibration

DISCLAIMER : these course notes are work in progress and still partially in French

You are welcome to send comments.

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Foreword

These notes are used for various courses given at Arts et Metiers ParisTech and Centrale-Supelec and in support of continuous education programs. They are **work in progress** and thus updated rather frequently with revisions now being made in English.

You can find them at <http://www.sdtools.com/pdf/PolyId.pdf>

For analytical aspects see chapters 2,5,6, 10.

For experimental modal analysis aspects see chapters 4, 7, 8, 9, 12.

Chapitre 1

Properties of single DOF systems

The simplest dynamic system considered in vibration problems is the Single Degree Of Freedom (SDOF) oscillator shown in figure 1.1. The analysis of this system is used to introduce the notions of transfer, poles, transient and forced response. It will also be shown in chap 3 that the response of more complex systems can be described as a sum of SDOF system contributions.

Assuming dissipation through a viscous damping mechanism the time response of this system is described by

$$m\ddot{q}(t) + c\dot{q}(t) + kq(t) = F(t) \quad (1.1)$$

relating the acceleration $m\ddot{q}$, dissipation $c\dot{q}$ and internal deformation kq forces to an external load $F(t)$.

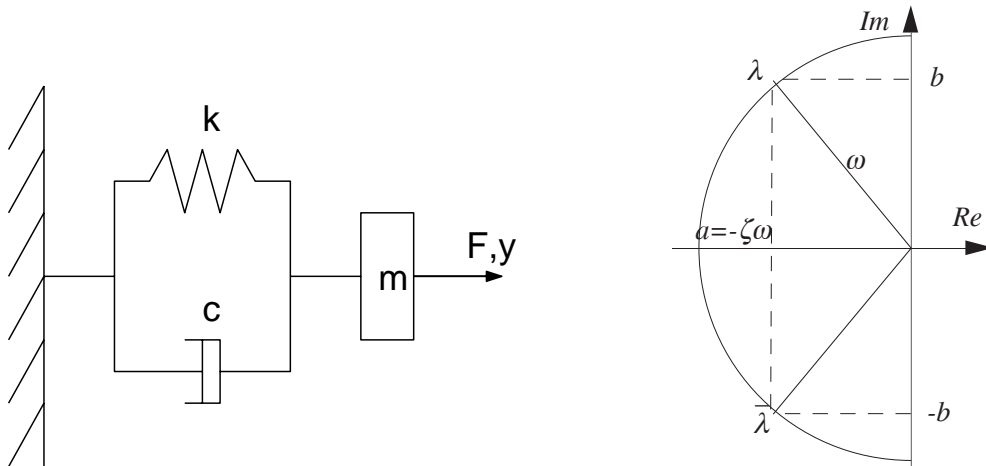


FIGURE 1.1 – One DOF model and associated poles

1.1 Transfer and poles

One considers here the permanent forced response to a sine excitation $F(t) = \text{Re}(F(\omega) e^{i\omega t})$ (also called harmonic input). It is a classical result that for a linear system the steady state

response to an harmonic input is also harmonic, that is $q(t) = \text{Re}(q(\omega) e^{i\omega t})$. By placing the known form of excitation and response in the dynamic equation of motion (1.1), one obtains

$$\text{Re} \left((-\omega^2 m + i\omega c + k) q(\omega) e^{i\omega t} - F(\omega) e^{i\omega t} \right) = 0 \quad (1.2)$$

From this one can extract the unique relation between $q(\omega)$ and $F(\omega)$ called *transfer function*

$$H(\omega) = \frac{q(\omega)}{F(\omega)} = \frac{1}{-\omega^2 m + i\omega c + k} = \frac{1}{m(-\omega^2 + 2i\zeta\omega_n\omega + \omega_n^2)} \quad (1.3)$$

which is equal to the Fourier transform of the *impulse response* $h(t)$.

The transfer function is a *rational fraction*. The roots of its denominator are called poles λ . For sub-critical damping ($c^2 < 4km$), they come as a complex conjugate pair

$$\begin{aligned} \lambda &= -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} \quad , \quad \omega_d = \omega_n\sqrt{1-\zeta^2} \\ \omega_n &= \sqrt{k/m} = |\lambda| \quad , \quad \zeta = \frac{c}{c_{crit}} = \frac{c}{2\sqrt{km}} = \frac{-\text{Re}(\lambda)}{|\lambda|} \end{aligned} \quad (1.4)$$

with a natural frequency ω_n and a damping ratio ζ (note that the damping ratio is also noted ξ or η depending country, community, ...).

For reference, it is useful to note that one calls *critical damping* the case where $\zeta = 1$, and that some industries use the notation $\zeta = \frac{c}{c_{critical}}$ even though this makes explicit reference to the one DOF mechanical model where as the definition of damping as the tangent of the pole is true for any dynamic system independently of the nature of the problem. By the same language abuse, people then call ζ viscous damping. Another common imprecise notion is the so called *hysteretic damping* $\eta = 2\zeta$ which corresponds to the notion of material loss factor and should not be used for dynamic systems [1].

The **quality factor** is a last common quantity used to define the damping ratio. For enforced based acceleration of a single mode system (equation (1.14)), the response at resonance is given by the so called quality factor (see (1.14))

$$Q_j = \frac{|x_{mass}|}{|x_{base}|} = \frac{1}{2\zeta_j} \quad (1.5)$$

For supercritical damping ($c^2 \geq 4km$), which is seldom found for vibrating structures, one has two real poles

$$\lambda = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m} - \frac{k}{m}} \quad (1.6)$$

The amplitude and phase of the transfer function $H(\omega)$ are shown in figure 1.2. One notes that the low frequency asymptote (given by $1/k$) corresponds to the static flexibility of the model (this quantity is meaningful for robot positioning for example) and the high frequency asymptote (given by $1/m\omega^2$) corresponds to the inertia contribution of the mass (the fact that it decreases is used for *isolation*).

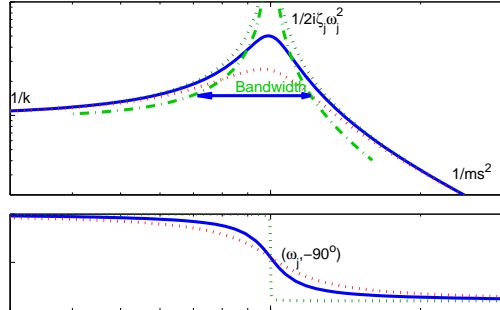


FIGURE 1.2 – Transfer function of the 1 DOF model

The **amplitude resonance** is the frequency of maximum response amplitude is obtained at $\omega = \omega_n \sqrt{1 - 2\zeta^2}$ with the transfer amplitude given by

$$|H(\omega)|^2 = \frac{1}{m^2 ((\omega_n^2 - \omega^2)^2 + 4\zeta^2 \omega_n^2 \omega^2)} \quad (1.7)$$

so that

$$|H(\omega_{max})| = \frac{1}{2\sqrt{2}m\omega_n^2\zeta\sqrt{1-\zeta^2}} \quad (1.8)$$

The maximum level of response to an harmonic excitation is the inversely proportional to the damping coefficient ζ which is often quite small (below 1%). This amplification phenomenon is known as *resonance*.

The resonance can also be defined as the point where the phase is equal to -90° (**phase resonance**) which somewhat easier to track experimentally. Phase resonance for a single DOF system is found at ω_n . Phase resonance is used as a mode indicator in experimental settings, see section 8.1.2, and for *non-linear normal mode testing*.

One can also show that the peak width (frequency separation of two points at 1/2 or -3dB the peak value) is equal to $2\zeta\omega_n$ for small damping levels. This width, as shown in figure 1.2, is thus directly proportional to the damping level. Modern versions of the -3 dB method are identification procedures, see section 7.2.

Equivalent bandwidth (integral of power spectrum) will be detailed here later.

1.2 Time response

The roots λ of the characteristic polynomial associated with the differential equation (1.1) are called poles. They correspond to the roots of the denominator associated with the transfer (1.3). The free response to a non-zero initial condition takes the general form

$$\begin{aligned}
 q(t) &= \operatorname{Re} \left(A e^{\lambda t} + B e^{\bar{\lambda} t} \right) = C \cos(\omega_d t + \phi) e^{-\zeta \omega_n t} \\
 &= e^{-\zeta \omega_n t} \left(q_0 \cos(\omega_d t) + \frac{\dot{q}_0 + \zeta \omega_n q_0}{\omega_d} \sin(\omega_d t) \right)
 \end{aligned} \tag{1.9}$$

Note intermediate steps $a \cos + b \sin$ derivation gives $\dot{q}_0 = -q_0 \zeta \omega_n + b \omega_n$. For supercritical damping ($c^2 \geq 4km$) the time response can be computed in a similar fashion with two purely decaying terms.

Figure 1.3 shows the time response to initial conditions $q(0) = 1$, $\dot{q}(0) = 0$ for various damping levels. One clearly sees the oscillatory behavior and the exponential attenuation of the response for subcritical damping. For critical damping the response goes to zero without oscillations. The return to zero is slower for higher than critical damping because one of the real poles has a lower frequency.

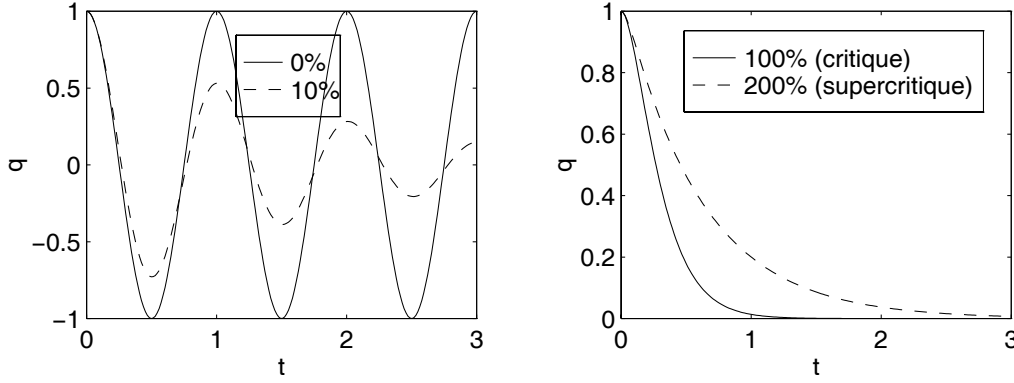


FIGURE 1.3 – Free decay of the 1 DOF model

Period and pseudo-period : for a damped exponential there is no period. $\omega_d = \omega_n \sqrt{1 - \zeta^2}$ is the frequency at which the response crosses 0 and is thus called a pseudo-period, while the natural frequency ω_n is associated with a period that cannot be measured directly in a time response.

The *Logarithmic decrement* is a classical approach for the estimation of damping. The ratio of two successive maxima q_n in the free response, leads to

$$\ln \frac{q_n}{q_{n+1}} = 2\pi\zeta \frac{\omega_n}{\omega_d} = \frac{2\pi\zeta}{\sqrt{1 - \zeta^2}} \tag{1.10}$$

One now studies the response to a step for input $F(t) = \delta(t \geq 0)$ and $q(t \leq 0) = 0$. The response can be decomposed into a steady-state response $q = F/k$ and a transient solution of the general form $(A \cos(\omega_d t + \phi) e^{-\zeta \omega_n t})$ that couples a oscillatory part and an exponential decrease. The figure below shows the response for various damping levels. One clearly see that for low damping levels the maximum response can be as high as twice the steady-state value (this is called the dynamic amplification factor).

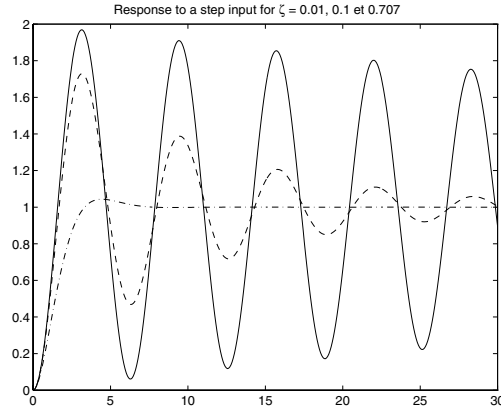


FIGURE 1.4 – Step response of the 1 DOF model

1.3 Classical extensions

1.3.1 Response to a moving base / isolation

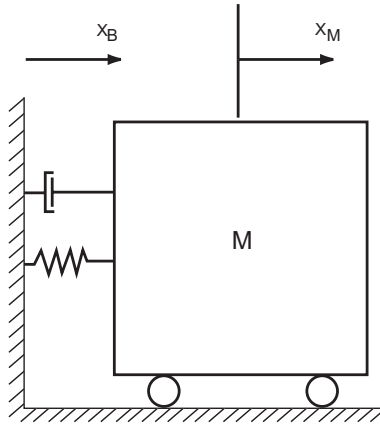


FIGURE 1.5 – 1 DOF system under inertial loading

One considers here the response of a single DOF system fixed to a reference frame with prescribed motion $x_B(t)$. The system equations can be described using absolute motion x_M of the mass

$$M\ddot{x}_M + C(\dot{x}_M - \dot{x}_B) + K(x_M - x_B) = F_M \quad (1.11)$$

which leads to

$$M\ddot{x}_M + C\dot{x}_M + Kx_M = F_M + C\dot{x}_B + Kx_B \quad (1.12)$$

or in a moving frame using the $x_R = x_M - x_B$ relative motion of the mass and the base

$$M\ddot{x}_R + Cx_R + Kx_R = M\ddot{x}_B + F_M \quad (1.13)$$

both equations describe different perspectives on the same system. One has the same characteristic equation $M\lambda^2 + C\lambda + K = 0$ and thus the same poles, but the forcing terms differ as apparent in the frequency domain transfer function representation

$$x_R = \frac{Ms^2}{Ms^2 + Cs + K}x_B = \frac{1}{M} \frac{s^2}{s^2 + 2\zeta_j\omega_j s + \omega_j^2}x_B \quad \text{and} \quad x_M = \frac{Cs + K}{Ms^2 + Cs + K}x_B \quad (1.14)$$

The transfer from x_B to x_M is called the transmissibility and is of particular interest since it represents the amplitude of suspended mass for a base acceleration. Vehicle or machinery suspensions follow derive from these fundamental characteristics.

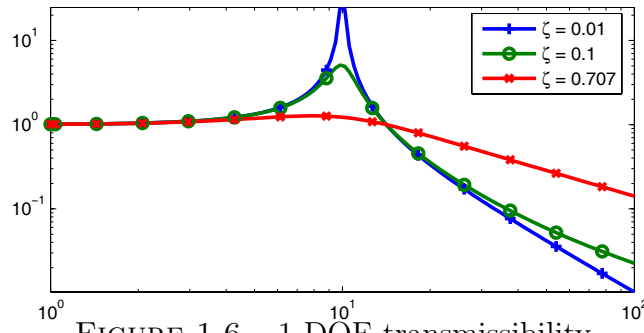


FIGURE 1.6 – 1 DOF transmissibility

1.3.2 Shock/response spectrum

For a given input $u(t)$, one computes (through numerical integration or through the use of a convolution (3.21) that is computed with an FFT and its inverse) the response of a unit mass oscillator that has poles $\lambda = -\zeta\omega \pm i\omega\sqrt{1 - \zeta^2}$

$$\ddot{y}_\omega + 2\zeta\omega \dot{y}_\omega + \omega^2 y_\omega = u(t) \quad \text{avec} \quad y(0) = 0 \quad (1.15)$$

At each oscillator frequency and damping, one defines the *shock/response spectrum* as being the maximum response of the preceding time integration

$$S(\omega, \zeta) = \max_t |y_\omega(t)| \quad (1.16)$$

The name used depends on the application area (seismology, aeronautics, ...) and the nature of the input (transient limited in time, stationary random input, ... You will find more details in [2, 3] and many other books.

One generally computes shock/response spectra for a frequency range of interest and a number of damping values that are representative of the considered structure.

Although spectra obtained for $\zeta = 0$ are often close to the amplitude of the Fourier transform of the shock, it should be clear that shock spectra are a univocal transform (one cannot determine the excitation based on knowledge of the spectrum).

Despite the fundamental difference between shock spectra and Fourier transforms, it is usual to define pseudo velocity spectra by $S_v(\omega) = \omega S_d(\omega)$ and pseudo acceleration by $S_a(\omega) =$

$\omega^2 S_d(\omega)$. For relatively low damping levels, the difference between pseudo and real spectra is often quite low.

In practice, the considered shocks are often idealized. One can also consider adimensionalized definitions ($\ddot{y}_{\max}/\ddot{u}_{\max}$, ...). Similarly various definitions of the maximum response can be retained. The primary spectrum gives the maximum response during the shock duration, the secondary spectrum the maximum after the shock, etc.

Finally, one should compute the spectrum associated of a quantity that is representative of the problem posed. Thus

- the absolute displacement can be used for a tolerance on a precision instrument
- the relative displacement will be more representative of internal deformations within the structure

Let us consider as an example the shock spectrum associated with a half sine input applied on the base of the seismic oscillator (figure 1.5). In the result shown in figure 1.7, one can distinguish

- the impulsive zone. For very low frequency oscillators (or very short sine) the shock has the effect of a force impulse. One generates a sudden change in speed.
- the static zone for very high frequency oscillators (or very long inputs) the behavior is constant and equal to the maximum of the enforced motion.
- an intermediate zone, where the dynamics of the oscillator can induce significant dynamic amplification.

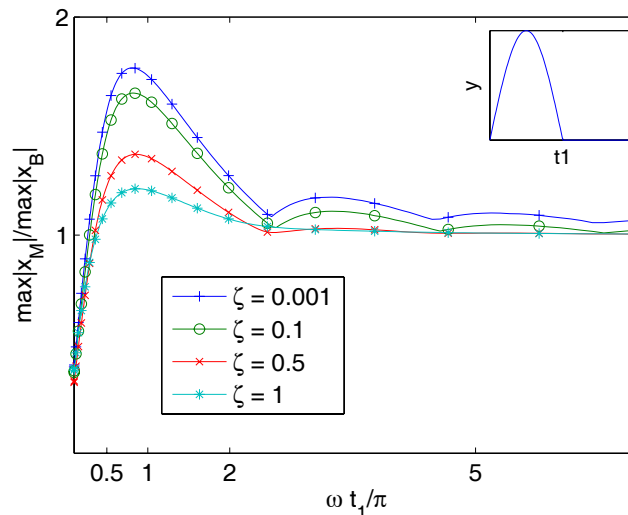


FIGURE 1.7 – Shock response of a seismic mass to a half sine

One will note in this plot that the best response is obtained for a maximum damping. A different conclusion is found if looking at response to steady-state inputs (see figure 1.6). As in most applications, this problem is thus the source of various trade-offs.

The characterization of a single maximum is not necessarily sufficient. For low cycle fatigue problems, one is interested in counting the number of times a certain threshold is reached. Figure 1.8 illustrates a possible representation of the probability of reaching various levels.

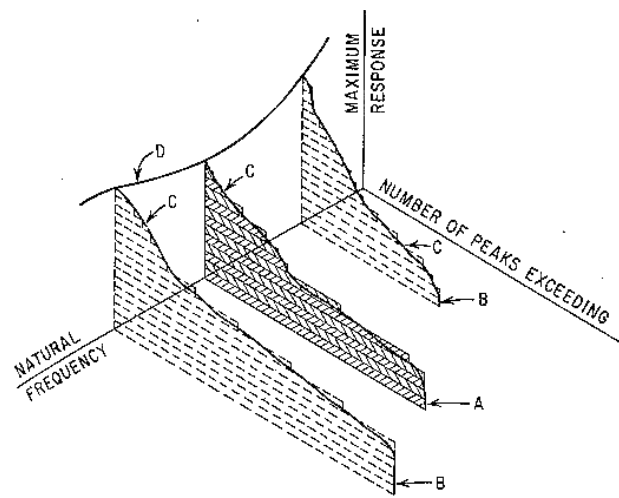


FIGURE 1.8 – Shock response histograms

Chapitre 2

Model derivation : continuous and discrete

2.1 Mathematical tools for model derivation

Need to write an intro here.

2.2 Kinematics (*cinématique*) : general and special cases

Continuous mechanics in large displacement assume that a point M of coordinates x_i have a displacement u_i with the original positions. Unknown fields in mechanical applications are displacements (3 components at each point). Mention multi-physics applications here.

Discuss domain Ω , Edge $\partial\Omega$

For **rigid body mechanics** : one assumes that motion is combination of translation and rotation. Noting x_i initial position, and X_i current position

$$\{u_i(x_i(M), t)\} = \{X_i(M) - x_i(M)\} = u_i(O, t) + ([R] - [I]) \{x_i\} \quad (2.1)$$

Classical parametrization of large rotations, discussed in [4] for example, is a combination of rotations around 3 axes. *Detail rotation in the xy plane and its linearization, then linearized expression of 3D rotation*

$$[[R] - [I]] \{x_i\} \approx \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} \{x_i\} = \{r_i\} \wedge \{x_i\} \quad (2.2)$$

which grouping the linear terms associated with the different linearized rotation components leads to

$$\begin{aligned} \{u_i(x_i(M), t)\} &= u_i(O, t) + r_i(t) \wedge OM = \begin{bmatrix} 1 & 0 & 0 & 0 & x_3 & -x_2 \\ 0 & 1 & 0 & -x_3 & 0 & x_1 \\ 0 & 0 & 1 & x_2 & -x_1 & 0 \end{bmatrix} \begin{Bmatrix} u_i(0, t) \\ r_i(t) \end{Bmatrix} \\ &= [T(x_i)] \{q(t)\} \end{aligned} \quad (2.3)$$

Need to draw a yz frame rotating around r_x to clarify how one checks signs.

In the last form (which assumes linearization), one performs a space $T(x)$ /time $q(t)$ decomposition with 6 unknowns : 3 translations and 3 rotations and 6 fields. *Note that later in the course $\{u \ v \ w\}$ may be used instead of $\{u_1 \ u_2 \ u_3\}$ to allow definition of field at different nodes.*

For beams (*poutres*), one adds the possibility of linear variations along one axis. The motion is thus that of a section at x_1 given.

$$u_i(x_i(M), t) = u_i(x_1, t) + OM \wedge r_i(x_1, t) = \begin{bmatrix} 1 & 0 & 0 & 0 & x_3 & -x_2 \\ 0 & 1 & 0 & -x_3 & 0 & 0 \\ 0 & 0 & 1 & x_2 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_i(x_1, t) \\ r_i(x_1, t) \end{Bmatrix} \quad (2.4)$$

Each “point” of the beam associated with the coordinate x_1 is a section (small surface perpendicular to the beam line). The unknown fields at any point of the line correspond to translation of the section center and rotations of the section.

For plates/shells (*plaques/coques*), one adds the possibility of variation along to coordinates of a surface that is flat for plates and curved for shells. The kinematics are now

$$u_i(x_i) = u_i(x_1, x_2) + OM \wedge r_i(x_1, x_2) = \begin{bmatrix} 1 & 0 & 0 & 0 & x_3 & 0 \\ 0 & 1 & 0 & -x_3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_i(x_1, x_2, t) \\ r_i(x_1, x_2, t) \end{Bmatrix} \quad (2.5)$$

Each “point” of the shell is a small segment perpendicular to the nominal surface. The unknown fields are now 3 translations of the section center and two rotations around directions in the plane tangent to the surface. The third rotation of around the normal has no physical meaning but is still used in FEM codes.

Rigid body mechanics leads to discrete models as there are a finite number of unknowns (6 per body minus some constraints). Beam, plates and continuous mechanics models are continuous and thus associated with an infinite number of unknowns. Exact continuous solutions are only known in a limited number of cases. In most practical applications, approximate solutions are sought within a finite subspace. To generate this finite subspace of possible solutions, one uses different tools

- Geometry decomposition. *Finite difference* methods use simple grid type domains : this is computationally efficient but does not easily account for complex geometries such as found in solid mechanics. *Finite element models* use simple geometries (lines, triangles, quadrangles, tetrahedra (triangular base pyramid), hexahedra (cube), pentahedral (hexahedron with two opposite faces degenerated to a triangle). Mapping can be used to generate non straight line/faces.
- Space/time decomposition within each domain. For any field that depends on space and time one assumes a decomposition in a series of functions of space and unknowns in time. These time coefficients are called DOF Degree Of Freedom (*DDL Degré De Liberté*)

Classic strategies to define a finite dimensional subspace two allow computer resolution are simple polynomials (finite element methodology), exponential in space (spectral methods based on continuous modes), other analytic functions that correspond to solutions of specific continuous mathematical problems (for example boundary element method).

2.3 Statics (*statique*) : loads / power

In all applications of physics, one introduces force (also called load, charge) fields dual to the displacement such that the product force-velocity is the power (*puissance*), or force-displacement increment is work (*travail*).

2.3.1 Rigid body statics

For rigid bodies, you have seen Newton equations (*principe fondamental de la dynamique*). The kinematics are described by 6 DOF (translations and rotations) and one defines 6 associated body loads (3 forces and 3 moments). The power is given by

$$P = \begin{Bmatrix} F_i \\ M_i(O, t) \end{Bmatrix}^T \begin{Bmatrix} \dot{u}_i(O, t) \\ \dot{r}_i(t) \end{Bmatrix} = \{\mathcal{F}_i\}^T \{\dot{q}_i\} \quad (2.6)$$

which can be rewritten as an integral over the volume (reestablish that continuous mechanics and rigid body mechanics coincide)

$$P = \int_{\Omega} \{f_j^v\}^T \{\dot{u}_j(x, t)\} = \int_{\Omega} \left(\{f_j^v\}^T \begin{bmatrix} 1 & 0 & 0 & 0 & x_3 & -x_2 \\ 0 & 1 & 0 & -x_3 & 0 & x_1 \\ 0 & 0 & 1 & x_2 & -x_1 & 0 \end{bmatrix} \right) \begin{Bmatrix} \dot{u}_i(O, t) \\ \dot{r}_i(t) \end{Bmatrix} \quad (2.7)$$

where one finds the usual definitions of rigid body forces and moment around the origin O , that is $\mathcal{F}_i(t) = \int_{\Omega} \{f_j^v(t)\}^T \{T_{ji}\}$.

As the work associated with body deformation is assumed to be negligible, the only internal loads being considered are associated with acceleration and described by an equation of the form

$$\begin{Bmatrix} F_{Acc} \\ M_{Acc} \end{Bmatrix} = \begin{Bmatrix} m\ddot{u} \\ I\ddot{r} \end{Bmatrix} = [M] \{\ddot{q}(t)\} \quad (2.8)$$

The **principle of virtual power** (*PPV principe des puissances virtuelles*) states that for the solution of equations of motion, total power is null. Total power is the combination of internal power which comes from forces associated with the system itself (elasticity, inertia, dissipation), and power that comes from external sources (inputs that can be force or displacement). Note that this is a knowledge model by opposition with a behavior model such as elasticity.

For a discrete model, verifying the PPV is fairly trivial. If you verify

$$[M] \{\ddot{q}(t)\} - \{F(t)\} = \{0\} \quad (2.9)$$

it will be verified in any basis. Thus for any test vector (choice of coordinate) $\{q^*\}$, one has

$$\{q^*\}^T [M] \{\ddot{q}(t)\} - \{q^*\}^T \{F(t)\} = \{0\} \quad (2.10)$$

You can easily see that this is a set of equations where each component has units of load/moment. Discuss the units of q^ : since for the true solution power is always zero, one can multiply q^* by a coefficient that may have any unit. Some will thus use the name *virtual work principle* assuming q^* to have unit displacement, while others will say *virtual power principle* (units of*

velocity) and mean the same thing. Integrating 0 power over time will also give work equal to 0.

It is useful to relate this formulation with the continuous model. The density of acceleration forces at any point of the body is given by $\rho \{\ddot{u}_i\}$, which given the space/time decomposition (2.3), has a spatial dependence of the form $\rho[T(x_i)] \{\ddot{q}(t)\}$. The relation between the body forces and the distributed field of acceleration forces is obtained by computing the work of acceleration forces for a “unit” displacement. Thus for acceleration along direction x_1 one uses the “virtual displacement”

$$T_1(x_i) = [T(x_i)] \begin{Bmatrix} 1 \\ 0 \\ \vdots \end{Bmatrix} = [T(x_i)] \{q_1^*\} \quad (2.11)$$

and computes the body force

$$F_{1Acc} = \int_{\Omega} [T_1(x_i) \rho T(x_i)] \{\ddot{q}\} = q_1^{*T} [M] \ddot{q} \quad (2.12)$$

The body force of rigid body mechanics is a “generalized” force (Craig [5] section 2.4). For the applied external loads, 3 forces and 3 moments, associated with each of the 6 unknowns describing the body motion, the power associated with a virtual field q^* is given by

$$P_{ext}(q^*) = \begin{Bmatrix} u_i^* \\ r_i^* \end{Bmatrix}^T \begin{Bmatrix} F_i \\ M_i \end{Bmatrix} = q^{*T} F \quad (2.13)$$

2.3.2 Flexible body : power computations in continuous mechanics

Within the volume of a beam in traction, one may have a density of axial force $N(x, t)$ (units of N/m). To compute the power density, you should remember for a beam theory class (*RDM résistance des matériaux*) that one uses the time derivative of axial strain u, x .

$$P = \int_{\Omega} N \frac{\partial \dot{u}}{\partial x} \quad (2.14)$$

Taking all 6 strain components of beams, the product of strain rate and associated stress components give power density at any location

$$P(x) = \{\tau\}^T \{\dot{\epsilon}\} = \begin{Bmatrix} N \\ T_y \\ T_z \\ M_t \\ M_y \\ M_z \end{Bmatrix}^T \frac{\partial}{\partial t} \begin{Bmatrix} u_{x,x} \\ u_{y,x} - r_z \\ u_{z,x} - r_y \\ r_{x,x} \\ r_{y,x} \\ r_{z,x} \end{Bmatrix} \begin{array}{l} \text{axial stiffness} \\ \text{transverse shear} \\ \text{transverse shear} \\ \text{torsion} \\ \text{bending} \\ \text{bending} \end{array} \quad (2.15)$$

Similarly for continuous mechanics in 3D, the linearized strain is given by

$$[\epsilon] = \frac{1}{2} (\nabla U + \nabla U^T) = \frac{1}{2} (u_{i,j} + u_{j,i}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (2.16)$$

and one introduces a stress tensor represented either as a 3×3 symmetric tensor/matrix or a 6×1 engineering stress vector. The power input into the structure is

$$P = \int_{\Omega} \sigma_{ij} \dot{\epsilon}_{ji} \quad (2.17)$$

where Einstein's convention of summing repeated indices is used.

2.4 Constitutive model, equations of motion

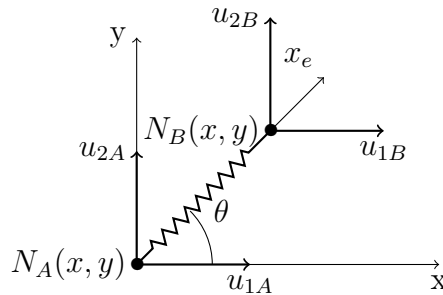
2.4.1 Constitutive laws : relation between loads and states

Kinematics define unknowns (finite number or field). Statics define "loads" so that power can be computed. To actually compute a response, you need to introduce a model of how loads depends on states. This is what you call a *constitutive model*. The simplest model form is a linear relation. You have seen this in

- rigid body mechanics : spring force proportional to elongation
- electricity : tension proportional to intensity in a resistance
- elasticity : stress proportional to strain

but you could consider a neural network found in artificial intelligence software which gives a very non-linear relation.

With the three legs : kinematics, statics and constitutive law you get **equations of motion**. Let us start with the example of a mass suspended on a spring undergoing large rotations in the xy plane.



$$\begin{aligned} \epsilon^e &= \{x_e\}^T \{\delta u\} = \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix}^T \begin{Bmatrix} u_{1B} - u_{1A} \\ u_{2B} - u_{2A} \end{Bmatrix} \\ &= [-1 \quad 1] \begin{Bmatrix} \{x_e\}^T \{u_A\} \\ \{x_e\}^T \{u_B\} \end{Bmatrix} = [-\{x_e\}^T \quad \{x_e\}^T] \begin{Bmatrix} \{u_A\} \\ \{u_B\} \end{Bmatrix} \\ &= [c] \{q\} \end{aligned} \quad (2.18)$$

The load associated is proportional to the local deformation : $F_e = k\epsilon^e$. The global load is then obtained using the fact that loads applied on by the springs on the two nodes must sum to zero. Thus

$$\{F\} = \begin{Bmatrix} -1 \{x_e\} \\ 1 \{x_e\} \end{Bmatrix} k\epsilon^e = [\{c^T\} k \{c\}] \{q\} \quad (2.19)$$

You *MUST* understand this. Used in ED 3-4. In class derivation of opposite forces from strain energy after Lagrange derivation.

In the case of a linear stress-strain relation $\{F\} = [K] \{q\}$, the energy stored in the system can be derived from a potential (called strain energy in mechanics). Indeed

$$E_k = \int_{[0, t]} (F^T \dot{q}) dt = \int_{[0, t]} (\{q\}^T [K] \{\dot{q}\}) dt = \frac{1}{2} \{q\}^T [K] \{q\} \quad (2.20)$$

Similarly, acceleration forces are assumed to be proportional to mass (density of continuous) and thus of the form $[M] \{\ddot{q}\}$, so that the work of these forces is given by

$$E_m = \int_{[0, t]} (F_a^T \dot{q}) dt = \int_{[0, t]} (\{\ddot{q}\}^T [M] \{\dot{q}\}) dt = \frac{1}{2} \{\dot{q}\}^T [M] \{\dot{q}\} \quad (2.21)$$

Discuss index : c for Cinétique (but kinetic in English), m for mass, k for stiffness, d for déformation (but strain in English).

Spring : example of a suspension block (bring physical object). Draw non-linear curve associated with hyperelasticity, what is the reference position ? (with or without weight), linearize around equilibrium position.

To induce motion you must spend some energy thus E_m and E_k are positive quantities.

Extension of virtual work to continuous case and link with Lagange's equations will be done in CM5 (strong formulation of modes) - CM6 (finite element modeling).

2.4.2 Equations of motion, Lagrange

For a system one defines a Lagrangian which describes the energy present in the system. The kinetic and strain energies can be used to describe the power/energy increments associated with inertia and elastic forces. Any other force (external, junction, dissipation, ...) will typically be considered external and the associated work will be noted $W_{ext}(q)$.

$$L(q, \dot{q}) = E_m(\dot{q}, q) - E_k(q) + W_{ext}(q, t) \quad (2.22)$$

and the Lagrange equations of motion are obtained using

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \left(\frac{\partial L}{\partial q} \right) = \{0\} \quad (2.23)$$

Note that during the derivation, q and \dot{q} are assumed to be independent variables (∂ not d). This derivation works even for large rotations and is thus used in Multibody software (MBS, for example Simpack or Adams).

Case large rotation spring with mass using l, θ parametrization. Things a simple for force (only one component) but much more complex for kinetic energy $E_m = \frac{1}{2}m(\dot{l}^2 + (l\dot{\theta})^2)$ the derivation of this kinetic energy has a component dependent on q .

$$\frac{d}{dt} \left\{ \begin{matrix} m\dot{l} \\ ml^2\dot{\theta} \end{matrix} \right\} - \left\{ \begin{matrix} 0 & -kl \\ m(\dot{\theta})^2l & -0 \end{matrix} \right\} = \{0\} \quad (2.24)$$

Case of external constant forces, one defines a potential as

$$W_{ext}(q) = \int_{[0, t]} (F_{ext}^T \dot{q}) dt = \{q\}^T \{F\} \quad (2.25)$$

Now you can verify that for the linearized case with expression of E_m and E_k given above, the equations of motion take the form

$$[M] \{\ddot{q}\} + [K] \{q\} = \{F\} \quad (2.26)$$

Discuss the fact that junction forces can be considered as either internal or external by just changing sign but need to be included.

Dissipation forces can be assumed

- dependent on velocity but independent of position (viscous model), this leads to $\{F_v\} = [C] \{\dot{q}\}$.
- dependent on position but independent of velocity (hysteretic model, also called friction or plasticity for different contexts), this requires internal states for time domain evaluation which will be discussed later.

2.4.3 Equations of motion, virtual power principle

Expressing the Lagrange equations for cases where forces do not derive from a potential (e.g. plasticity) can be more tricky, one then prefers another methodology. For the true forces associated with a trajectory, the power associated with an arbitrary velocity field (called *virtual*) will be equal to 0. The conventional choices of signs lead to

$$P_{Acc}^* = P_{Int}^* + P_{Ext}^* + P_{Junction}^* \quad (2.27)$$

For E_m kinetic energy, q displacement, \dot{q} velocity, one has

$$P_{Acc}^* = \{q^*\}^T \frac{d}{dt} \left(\frac{\partial E_m}{\partial \dot{q}} \right) = \int_{\Omega} (\{u^*\}^T \frac{d^2}{dt^2} u(x, y, z, t)) \quad (2.28)$$

For E_k strain energy (note that putting a minus sign in the Lagrange equations is non an obvious convention)

$$P_{Int}^* = - \{q^*\}^T \frac{\partial E_k}{\partial q} = \int_{\Omega} (\{\epsilon(u^*)\}^T \sigma(x, y, z, t)) \quad (2.29)$$

For a continuous system, forces are distributed and dependent on orientation (stress). Power computations were discussed earlier and will be reused for weak form derivation and finite element modeling.

For external forces, it is important to distinguish the spatial and temporal dependence of loads. One will thus write

$$\{F_i(t)\} = [b_i]_{N \times N_A} \{u(t)\}_{N_A} = \left[\int_{\Omega} \{T_i(x)\}^T \{b(x)\} \right] \{u(t)\} \quad (2.30)$$

Thus for the case of a point load where $b(x) = \delta(x - x_P)$, one has $\{F_i(t)\} = \{T_i(x_P)\} \{u(t)\}$. b_i corresponds to what is called a command matrix in state space models.

2.5 Modes of continuous systems

2.5.1 Weak and strong form

The equations of motion derived from the virtual power principle (2.27) is, in applied mathematics, called a weak form. This refers to the fact that the equilibrium is verified an integral sense for every vector in a subspace and not by a function of every point in space. The later is a strong or local form using a set of PDE (partial differential equations).

Taking the example of beam compression in statics

$$P_{Int}^* = - \int_{\Omega=[0 \ L]} \epsilon^* A \sigma_{xx} = - \int_{\Omega} \frac{\partial u^*}{\partial x} EA \frac{\partial u}{\partial x} \quad (2.31)$$

where $N = EA \frac{\partial u}{\partial x}$ is the normal load. This can be integrated by parts leading to

$$P_{Int}^* = - [u^* EA \frac{\partial u}{\partial x}]_{x=0}^L + \int_{\Omega} u^* \frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x} \right) \quad (2.32)$$

Since this is true for any kinematically admissible function (continuous function) $u^*(x)$, the strong form of equations of motion are defined

- within the domain Ω , one verifies the PDE $\rho A \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} (EA \frac{\partial u}{\partial x}) = f_v(x)$. This requires
 - a description of the domain (typically FEM geometry)
 - a problem formulation (you have seen : compression, bending, torsion, continuous mechanics = 3D solid, possibly plate/shell)
 - constitutive parameters to represent the material
- on the edge of the spatial domain $\partial\Omega$, one verifies *boundary conditions* in displacement (clamped), force (free or point load), or combination (added spring). You should remember that boundary conditions provide sufficient data to compute power flow : displacement or force but not both. Or a function that gives force as a function of displacement (constitutive law on edge).
- on the edge of the time domain, one talks about *initial conditions*.

Thus one can compare the equations needed in the strong and weak forms

	strong/local	weak/PVW
On Ω	$\rho A \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} (EA \frac{\partial u}{\partial x}) = f_v(x)$	$\int_{\Omega} (\rho A u^* \ddot{u} + u^*_{,x} EA u_{,x} - u^* f_v) = 0$
On $\partial\Omega$		
- Right fixed	$u(x_R) = 0$	$u(x_R) = 0$
- Right free	$F_x = EA u_{,x} _{x_R} = 0$	- (not needed since no work)
- Right loaded	$F_x = EA u_{,x} _{x_R} = F_{ext}$	$P_{ext} = [b] F_{ext}(t) = u^*(x_R) F_{ext}(t)$
- Spring at right	$EA u_{,x}(L) - ku(L) = 0$	$\int_{\Omega} \dots + ku(L) u(L) = F_{ext}$

Note that in the case of a forced response in the weak form, the local observation of the right force $F_x = EA u_{,x}$ may not be equal to the external force (this will be addressed again in the FEM section). *Check that you know how to deal with local dashpot, local mass, periodic solution.*

The equations for beam bending in local and integral (principle of virtual work) form

	local	PVW
On Ω	$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial w^4}{\partial x^4} = f_v(x)$	$\int_{\Omega} \rho A \hat{w} \ddot{w} + EI \hat{w}_{,x^2} w_{,x^2} = F_{ext}$
On $\partial\Omega$		
Right fixed	$w(x_R) = 0, w_{,x} = 0$	$w(x_R) = 0, w_{,x} = 0$
Right pinned	$w(x_R) = 0, EI w_{,x^2} = 0$	$w(x_R) = 0$
Right free	$M_y = EI w_{,x^2} = 0, F_z = -EI w_{,x^3} = 0$	– (no load needed)
Spring	$EI w_{,x^2} = 0, EI w_{,x^3}(x) - K w(x) = 0$	$\int_{\Omega} \dots + K \hat{w}(x) w(x) = F_{ext}$

Check that you know how to deal with local torsion spring, mass, dashpot.

The last classical continuous model is that of plate bending

$$D \Delta \Delta w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 = D \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + \rho h \frac{\partial^2 w}{\partial t^2} \quad (2.33)$$

with $D = \frac{Eh^3}{12(1-\nu^2)}$. For composites, [6, 7] are classical reference texts.

The few problems that can be solved exactly in the strong form are useful to build analytical references, but the majority of applications is in approximate solutions (see the FE modeling section 2.6).

2.5.2 Frequency /wavenumber domain resolution in strong form

To obtain analytical solutions, one generally assumes the system to be linear and time invariant and seeks to evaluate the frequency domain response. Taking the Fourier transform (Laplace could also be used), one thus seeks forced responses of the form $\{u(x, t)\} = \{u(x)\} e^{i\omega t}$. The forced harmonic response verifies a differential equation in space

$$\rho A \omega^2 u(x) + EA \frac{\partial^2 u(x)}{\partial x^2} + f_v(x) = 0 \quad (2.34)$$

When the coefficients are constant in space and there are no distributed external loads ($f_v = 0$), the solutions in space of this differential equation are a linear combination of harmonic functions of space, called *waves (ondes)*,

$$\{u(x, \omega)\} = \sum_k \{e^{\kappa_k(\omega) x}\} a_k(\omega) \quad (2.35)$$

Inserting this expression into the differential equation leads to

$$\sum_k (a_k(\omega) e^{\kappa_k(\omega) x} (\rho A \omega^2 + EA \kappa_k^2)) = 0 \quad (2.36)$$

Since this equation is verified for all x , one has an equation relating frequency and wavenumber, called the *dispersion equation*,

$$\rho A \omega^2 + EA \kappa_k^2 = 0 \Rightarrow \kappa_k = \pm i \omega \sqrt{\frac{\rho}{E}} \quad (2.37)$$

The dispersion equation ensures that for solutions of the form (2.36) equations of motion are verified within the domain. The next step is to account for boundary conditions. Typical boundary conditions were listed in section 2.5.1.

For a clamped free beam with a load applied on the edge. The edges of the domain $\partial\Omega$ are $x = 0$ or $x = L$, and one verifies $u(x) = 0$, and $EA\frac{\partial u(L)}{\partial x} = F_{ext}$. Combining these boundary conditions with the expression (2.36) of the solution as a combination of waves leads to a linear problem in the wave amplitudes

$$\begin{bmatrix} 1 & 1 \\ k_1 EAe^{k_1 L} & k_2 EAe^{k_2 L} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_{ext} \end{Bmatrix} \quad (2.38)$$

In this case the wave numbers are purely imaginary and conjugate $\kappa_k = \pm i\omega\sqrt{\frac{\rho}{E}}$. It is thus common to combine the exponentials and rewrite the solution as $u(x, \omega) = a_1 \cos(\hat{k}x) + a_2 \sin(\hat{k}x)$ with $\hat{k} = \omega\sqrt{\frac{\rho}{E}}$ leading to an equivalent problem

$$\begin{bmatrix} \cos(0) & \sin(0) \\ -\hat{k}EA\sin(\hat{k}x) & \hat{k}EA\cos(\hat{k}x) \end{bmatrix} \begin{Bmatrix} a_1(\omega) \\ a_2(\omega) \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_{ext} \end{Bmatrix} \quad (2.39)$$

In class show some illustration of dispersion equation for catenary [8] or honeycomb [9]. Mention boundary elements which use analytical solutions within the domain.

2.5.3 Analytic expression of continuous modes

Modes are a combination of waves that also verify time independent boundary conditions on the edge $\partial\Omega$ in the absence of external excitation (harmonic solution in the absence of excitation). Because the boundary conditions do not change, the modes are also sometimes called *standing waves*.

Since they are harmonic in frequency, one reuses the frequency domain forced response problem but now with a null right hand side (no forcing)

$$\begin{bmatrix} 1 & 0 \\ -EA\hat{\kappa}\sin(\hat{\kappa}L) & EA\hat{\kappa}\cos(\hat{\kappa}L) \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (2.40)$$

and to obtain modal frequencies, one seeks non zero roots of this problem (frequencies for which the determinant is equal to zero).

For the case of traction, coming back to the expression of the forced response (2.39), the waves amplitudes can be non zero in the absence of loading (right hand side of the equation) if the matrix is singular. In this trivial example, one thus needs $\cos(\hat{k}L) = 0$ or

$$\hat{k}L = \pi(j + \frac{1}{2}) = i\omega_j L \sqrt{\frac{\rho}{E}} \quad \omega_j = (j + \frac{1}{2}) \frac{\pi}{L} \sqrt{\frac{E}{\rho}} \quad (2.41)$$

Note that in the case of undamped mechanical systems, the poles are purely imaginary $\lambda_j = i\omega_j$, whereas for damped system, one must choose between computing the forced response a purely imaginary frequency $s = i\omega$ or computing the periodic response in space at a wavenumber $k = i\hat{k}$.

The mode shapes are solution of for cases where the problem is singular. Here there the first row of (2.5.3) gives $a_1 = 0$ and thus the modes are of the form

$$\phi_j(x) = a_2 \sin(\hat{k}x) = \alpha_j \sin(\hat{k}x) \quad (2.42)$$

where the scaling coefficient α_j should be such that the result is mass normalized. In other words

$$\mu_j = 1 = \int_{\Omega=[0 \quad L]} \phi_j(x) \rho A \phi_j(x) = \rho A \alpha_j^2 \int_{\Omega} \sin^2\left(\pi \frac{x}{L} \left(j + \frac{1}{2}\right)\right) \quad (2.43)$$

leading to

$$\alpha_j = \sqrt{\frac{1}{\rho A L \int_0^L \sin^2(\pi r (j + \frac{1}{2})) dr}} \quad (2.44)$$

You should note the classical result that $\int_0^L \sin^2(j\pi \frac{x}{L}) dx = \frac{L}{2}$ which is the case where mass normalisation is simple (clamped/clamped traction, simply supported beam, simply supported plate, ...).

Once the modes mass normalized, it is possible to compute the modal commandabilities (3.3) and observabilities as in (3.4) for discrete models. For a point load applied on point x_F , the modal force is given by

$$\{F_j(t)\} = \{b_j\} f(t) = \left\{ \int_{\Omega} \phi_j(x) \delta(x - x_F) \right\} f(t) = \{\phi_j(x_F)\} f(t) \quad (2.45)$$

Similarly for observation, the displacement at point x_F is given by

$$u(x, t) = \sum_j c_j q_j(t) = \sum_j \phi_j(x_F) q_j(t) \quad (2.46)$$

In the end for continuous modes the general form of a modal model is still(3.45)

$$H(\omega) = \sum_{j=1}^{\infty} \frac{[c] \{\phi_j\} \{\phi_j\}^T [b]}{-\omega^2 + 2\zeta_j \omega_j i \omega + \omega_j^2} \quad (2.47)$$

In practice, the approximation comes from the fact that the series must be truncated for practical evaluation. The nominal strategy combines modes and a constant approximation of the rest of the series called static correction [10].

The case of beam bending will be done in TD 5-6. If a little time in class, do simply supported plate bending.

Modes of continuous systems can be computed analytically for a few classical configurations. It is useful to know the procedure for the partial differential equations (PDE) with constant coefficient found for the beam and 2D plate equations. Other configurations can be found in books. [11] is a classical reference.

2.6 Ritz approximation, weak form, FEM models

Classical formulations are summarized in section A.5.

2.6.1 FEM geometry and displacement

As detailed in section 2.2, the first step of model derivation is a description of kinematics. For complex geometries, the finite element method starts by describing a structure as a set of *elements* geometrically simple geometries

- points (point masses, springs connected to ground or connecting points)
- lines associated with mechanical models of rod (traction/compression, up to 3 translations/node) or beams (traction, bending, torsion, up to 3 translations and 3 rotations per node)
- surfaces : plates (if flat), shells (if curved), contact surfaces
- volumes

Associated geometries are well documented, in classical FEM books [12, 13, 14] and on-line resources like the Abaqus manual or Code Aster (<https://www.code-aster.org/doc/default/en/>).

Figure 2.1 illustrates classical node numbering used to describe the geometry.

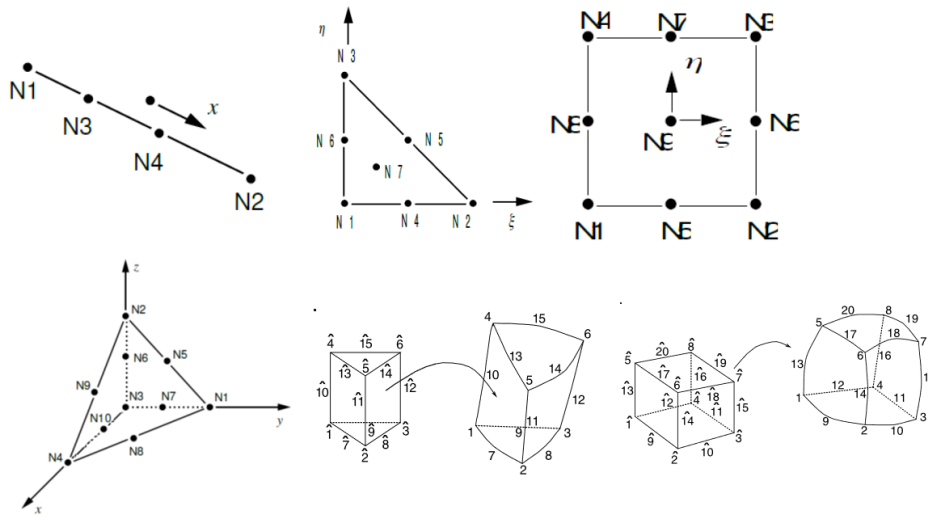


FIGURE 2.1 – Positions of nodes for classical geometries [15]. 1D : lines, 2D : triangles, quadrangles, 3D : tetrahedra, pentahedra, hexahedra

Within each geometric entities, one usually makes the following assumptions

- each unknown field is described using space/time decomposition $f(x) = \sum_i N_i(x)q_i(t)$. Functions of space are called shape functions and will be noted $N(X)$, unknown in time are called Degree Of Freedom.
- shape functions are taken to be polynomial functions of space
- shape functions are generally chosen so that their value is equal to 1 at a single node so that for any field u , one can write

$$\{u(x, t)\} = \sum_i \{N_i(x)\} u_i(t). \tag{2.48}$$

- rather than using a global coordinate system, one generally defines a reference geometry and a geometry mapping for each element. Thus the X position

$$\{X(x)\} = \sum_i \{N_i(r)\} X_i \tag{2.49}$$

— usual verifications are the ability to represent rigid body motion and constant strain states exactly.

For a 2 node line (beam in traction-compression), taking the reference segment to be $r = \frac{x-x_0}{L} \in [0 \ 1]$, the two shape functions are

$$N_1(r) = 1 - r \text{ and } N_2(r) = r \quad (2.50)$$

which clearly verify $N_1(0) = N_2(1) = 1$ and $N_2(0) = N_1(1) = 0$.

The case of cubic shape functions associated with beams is treated in ED 6.

2.6.2 Strain

Strain is obtained by deriving the shape functions with respect so space (since the DOF do not depend on space, they do no need to be derived). As a geometry mapping is typically used, one may need to use the derivative of (2.49). Thus for the case of lines

$$\{\epsilon(x, t)\} = \sum_i \frac{\partial N_i(x)}{\partial x} u_i(t) = \sum_i \frac{\partial r}{\partial x} \frac{\partial N_i(r)}{\partial r} u_i(t) \quad (2.51)$$

One typically calls Jacobian the derivative of position with respect to reference coordinates. Taking the case of a mapping to x, y global coordinates from r, s reference element coordinates.

$$\left\{ \begin{array}{c} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial s} \end{array} \right\} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \left\{ \begin{array}{c} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{array} \right\} = J \left\{ \begin{array}{c} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{array} \right\} \quad (2.52)$$

Since derivatives with respect to the reference coordinates $\frac{\partial N_i}{\partial r}, \dots$ are constant, the need for a mapped element is to compute the inverse of the Jacobian

$$\left\{ \begin{array}{c} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{array} \right\} = J^{-1} \left\{ \begin{array}{c} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial s} \end{array} \right\} \quad (2.53)$$

In the case of the a line with two nodes on the interval $\Omega = [0 \ 1]$, $x = (1 - r)x_1 + rx_2$, thus $J(r) = \frac{\partial x}{\partial r} = x_2 - x_1 = L$.

2.6.3 Equations of motion weak form/virtual power

For the principle of virtual work (2.27), one seeks the solution within a subspace of the cinematically admissible displacements and proves that it is such that the work of this solution is zero for all virtual displacements within the same subspace.

For the beam in traction compression, one thus has

$$\int_0^L \frac{\partial \hat{u}}{\partial x} EA \frac{\partial u}{\partial x} + \int_0^L \hat{u} \rho A \frac{\partial^2 u}{\partial t^2} = \int_0^L \hat{u} f_v(x) \quad (2.54)$$

with the integrals giving the strain, kinetic and external work contributions. A Ritz approximation (also called assumed modes method) is obtained by assuming $u(x, t)$ and $\hat{u}(x)$ to be within the subspace generated by a set of functions $T_i(x)$, that is $u(x, t) = \sum_i T_i(x) q_i(t)$ (in

the case of FEM models, the shape function within an element $N_i(x)$ is the restriction of the global shape $T_i(x)$ to this element, and polynomials on each element is the usual case). Since u and \hat{u} appear linearly in the strain and kinetic energies one has

$$\{\hat{q}\}^T [K] \{q\} = \int_0^L \frac{\partial \hat{u}}{\partial x} EA \frac{\partial u}{\partial x} \quad \text{thus} \quad K_{ij} = \int_0^L \frac{\partial T_i}{\partial x} EA \frac{\partial T_j}{\partial x} \quad (2.55)$$

Similarly

$$\{\hat{q}\}^T [M] \{\ddot{q}\} = \int_0^L \hat{u} \rho A \frac{\partial^2 u}{\partial t^2} \quad \text{thus} \quad M_{ij} = \int_0^L T_i \rho A T_j \quad (2.56)$$

Finally

$$\{\hat{q}\}^T \{b\} = \int_0^L \hat{u} f_v(x) \quad \text{thus} \quad b_i = \int_0^L T_i f_v(x) \quad (2.57)$$

The input matrix b corresponds to the force associated to a unit load. One assumes distributed loads to be of the form $f_v(x, t) = \sum f_{vk}(x) u_k(t)$, where f_{vk} are unit loads associated with inputs u_k . One thus has

$$b_{ik} = \int_0^L T_i f_{vk}(x) \quad (2.58)$$

The observation matrix c is simply derived from the kinematic description that uses variable separation $u(x, t) = \sum_i T_i(x) q_i(t)$. Thus coefficients of the observation matrix for a displacement at x_0 is given by

$$c_i = T_i(x_0) \quad (2.59)$$

and for a strain ϵ_{xx}

$$c_i = \left. \frac{\partial T_i}{\partial x} \right|_{x_0} \quad (2.60)$$

In this form, one only considers boundary conditions on the displacement (those that make the solution, cinematically admissible). Thus a clamped end at $x = 0$ leads to condition $u(0) = 0$, but a free end need not be considered. A spring support at the end however leads to energy, and thus needs to be incorporated as a local contribution $k\hat{u}u$.

2.6.4 Computing integrals, numerical quadrature, Gauss points

Taking again our beam example, one can seek a solution within the subspace of second order polynomials $u(x, t) = \hat{q}_0 1 + \hat{q}_1 x + \hat{q}_2 x^2$. 1 , x , and x^2 are basis vectors (called T_i elsewhere in these notes). $\{\hat{q}_i\}$ are called degree of freedom (DOF). Rather than using the base polynomial, one seeks a combination of these functions that verifies $N_i(x_j) = \delta_{ij}$ for three equally spaced nodes $x_1 = 0, x_2 = L/2, x_3 = L$ and a reference geometry $r \in [0 \ 1]$. You can verify that

$$u(r) = \begin{bmatrix} 2(r-1)(r-.5) & 4r(1-r) & 2r(r-.5) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \quad (2.61)$$

Note how each shape function is the product of 2 polynomials with roots at the other nodes (r zero at x_1 , $1 - r$ zero at x_3 , $r - .5$ zero at x_2).

Computing the integrals (2.55), ... can be done explicitly for beams, but often not with mapped geometries where a jacobian (2.52) is needed. The normal procedure is thus to use numerical quadrature (seek *gaussian quadrature* on Wikipedia if you don't remember). Thus an integral is approximated as a series

$$\int_{\Omega} f(x) = \sum_g w_g \det(J_g) f(x_g) \quad (2.62)$$

where w_g is the gauss weight (you can think of it as the volume associated with a given gauss point in the reference geometry), J_g is the Jacobian introduced in (2.52) (this gives the relation between volume in the true (mapped) geometry and volume in the reference one) and $f(x_g)$ is the evaluation of the function to be integrated at the Gauss point x_g . In general it is not necessary to evaluate the position of X_g of the Gauss points since the Gauss points positions, shape functions and their derivatives are known in the coordinate system of the reference element (see slide discussing MATLAB/SDT format).

It is important to understand that since stress is only computed at Gauss points, evaluations elsewhere may be quite wrong. This is illustrated in the slides, where one shows the approximation of a sine function using linear shape functions.

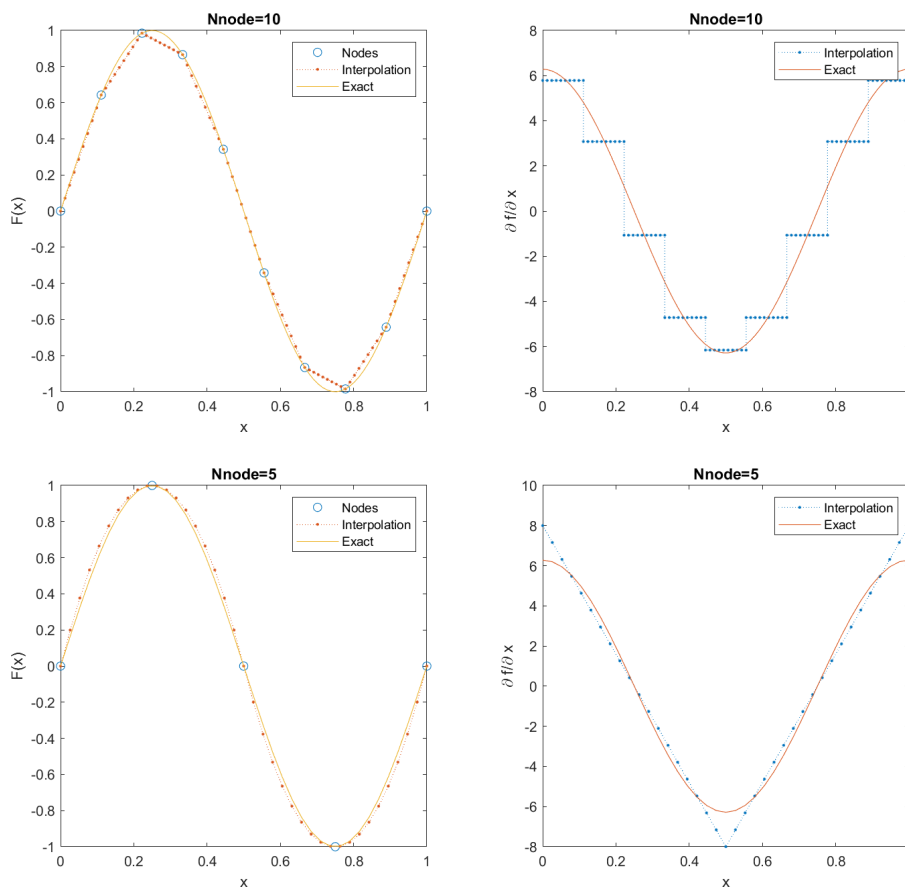


FIGURE 2.2 – Approximation of a sine wave and its derivative using linear and quadratic shapes

2.6.5 Numerical issues

Things addressed in slides <https://savoir.ensam.eu/moodle/mod/folder/view.php?id=47532> file [CM6_FEM.pdf](#) and the MATLAB demo are

- connection and sparsity
- ordering an computation time
- factorization and resolution. : why direct frequency response is stupid.
- issues associated with parallelism

Chapitre 3

System models, DOF, states, ...

This chapter considers the dynamic properties of linear systems with multiple degrees of freedom (MDOF). The presentation voluntarily seeks to put the mechanical and control perspectives in parallel. For a classical presentation from the mechanical point of view, Ref. [16] is good.

Section 3.1 presents the relation between state-space and mechanical equations. A short reminder of Ritz approximations is used to illustrate the relation between states and DOFs. To illustrate uses of state-space models, time response is addressed through exact solutions with the matrix exponential and approximate time integration schemes. Frequency domain solutions and the notion of transfer function are finally addressed.

Section 3.2 discusses elastic, or normal, modes which are the base ingredient for mechanical applications. Section 3.3 discusses modes in the more general state space form. Finally, section 3.5 gives equations for the case of seismic responses which is a second important problem for mechanics applications.

3.1 State space and mechanical forms

3.1.1 General model forms, input/output shape matrices

With very few restrictions, the evolution of linear dynamic systems can be represented using first order linear differential equations, called *state-space models*, of the form

$$\begin{aligned}\{\dot{x}(t)\} &= [A] \{x(t)\} + [B] \{u(t)\} \\ \{y(t)\} &= [C] \{x(t)\} + [D] \{u(t)\}\end{aligned}\tag{3.1}$$

This general form is supported by most simulation packages (Matlab/Simulink, Adams, ...) and many theoretical developments in control theory. As shown in the block-diagram of figure 3.1, the system is characterized by the relation between inputs u and outputs y .

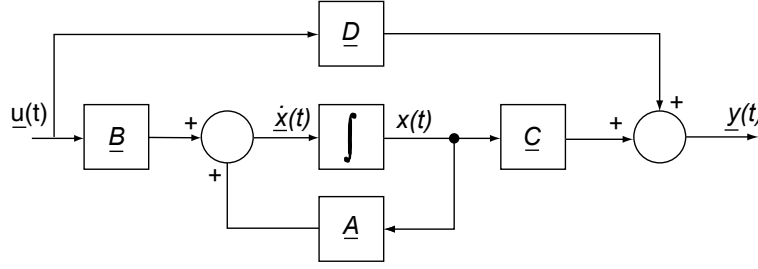


FIGURE 3.1 – Block-diagram representation of a state-space model

For mechanical systems, the use of an approximation method (based on the principle of virtual works, Lagrange's equations, the finite element method, etc., see details in later sections) leads to equations of motion of the general form

$$[M] \{\ddot{q}(t)\} + [C] \{\dot{q}(t)\} + [K] \{q\} = \{F(t)\} \quad (3.2)$$

Although mechanical engineers tend to think that *Degrees of Freedom* $\{q\}$ are physical because they represent motion at a particular point, they are really just coefficients describing the continuous motion as a linear combination of continuous shapes (see chapter 5) and it is quite important for structural dynamics to use other **generalized** DOFs than responses at particular nodes.

To allow this change, one must complement (3.2) by defining inputs and physical responses in a manner that is independent of the choice of $\{q\}$. As in control theory, one decomposes the applied loads

$$\{F(t)\} = [b] \{u(t)\} \quad (3.3)$$

as the product of a **input shape matrix** $[b]$ that is time independent and a vector of inputs $\{u(t)\}$ characterizing the time or frequency dependence of the load.

Similarly, one defines physical outputs $\{y\}$ by an **observation equation**

$$\{y(t)\} = [c] \{q(t)\} \quad (3.4)$$

which relates DOFs q (the equivalent of states X in state-space models) to outputs. Typical mechanical outputs are displacements, strains, stresses (possibly non-linear function of stresses), resultant forces.

In class, the continuous beam in traction/compression is used to illustrate this concept. One insists on the fact that decomposing in space and time is needed in all software packages. Images are shown to illustrate input at FEM DOF and in between.

The use of second order equations is not fundamentally different from the resolution using state-space equations. For example, one can rewrite (3.30) as the canonical state-space model

$$\begin{aligned} \begin{Bmatrix} \dot{q} \\ \ddot{q} \end{Bmatrix} &= \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} + \begin{bmatrix} 0 \\ M^{-1}b \end{bmatrix} \{u\} \\ \{y\} &= [c \ 0] \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} \end{aligned} \quad (3.5)$$

Although often cited, this particular form is a **very poor** choice. A few reasons are : for FEM models, M can often not be inverted ; in modal coordinates (see section 3.2), the mass is identity and thus does not need to be inverted ; the realization is not balanced.

For computations of complex modes in state-space form, this particular representation also loses properties of symmetry that are present in the initial matrices. One thus sometimes considers the generalized state space model

$$\begin{aligned} \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{Bmatrix} \dot{q} \\ \ddot{q} \end{Bmatrix} + \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} &= \begin{bmatrix} b \\ 0 \end{bmatrix} \{u\} \\ \{y\} &= \begin{bmatrix} c & 0 \end{bmatrix} \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} \end{aligned} \quad (3.6)$$

The two forms (3.5)-(3.6) are not very appropriate for state space integration, it will be shown that normal modes lead to the really useful state-space form (5.21).

3.1.2 Ritz approximation of continuous systems

Ritz-Galerkin methods can be considered as a specific case of model reduction methods that are addressed in chapter 5. They are however typically taught as the underlying basis for finite element modeling techniques. The objective of this section is thus to give an example of how the matrices mentioned in the previous section can be built. The case of a beam in traction / compression is used to relate continuous mechanics to the second order models and the section serves as a reminder for topics usually taught in a different class.

Continuous mechanics can be described in strong or local form using a set of PDE (partial differential equations) defined within volumes, boundary conditions defined on the spatial edges and for transient problems initial conditions defined on the time edges. Classical solutions are summarized in section A.5.

For a beam in traction, the kinematics of the motion are described by the axial displacement $u(x)$. The stress equilibrium is given by

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{f_v(x)}{A} = \rho \frac{\partial^2 u}{\partial t^2} \quad (3.7)$$

Assuming an elastic behavior, the stress/strain relation is given by

$$\sigma_{xx}(x) = E \epsilon_{xx}(x) = E \frac{\partial u}{\partial x} \quad (3.8)$$

The classical local equation for the beam, combines the stress equilibrium and the constitutive equation of elasticity, to give

$$EA \frac{\partial^2 u}{\partial x^2} + f_v(x) = \rho A \frac{\partial^2 u}{\partial t^2} \quad (3.9)$$

Boundary conditions are given in displacement $u(x) = 0$, stress $E \frac{\partial u}{\partial x} = 0$ or combination in the case of end springs.

Initial conditions are given in displacement and velocity.

The few problems that can be solved exactly in the strong form are useful to build analytical references, but the majority of applications is in approximate solutions. These approximations are introduced in the literature using many different names : the principle of virtual work, the weak form of the PDE, the Lagrange equations, ... All these methods are related and it is not the point of these notes to try highlighting their differences. REFERENCE TO SCHOOL SPECIFIC COURSES SHOULD BE GIVEN HERE.

For the principle of virtual work, one seeks the solution within a subspace of the cinematically admissible displacements and proves that it is such that the work of this solution is zero for all virtual displacements within the same subspace.

For the beam in traction compression, one thus has

$$\int_0^L \frac{\partial \hat{u}}{\partial x} EA \frac{\partial u}{\partial x} + \int_0^L \hat{u} \rho A \frac{\partial^2 u}{\partial t^2} = \int_0^L \hat{u} f_v(x) \quad (3.10)$$

with the integrals giving the strain, kinetic and external work contributions.

In this form, one only considers boundary conditions on the displacement (those that make the solution, cinematically admissible). Thus a clamped end at $x = 0$ leads to condition $u(0) = 0$, but a free end need not be considered. A spring support at the end however leads to energy, and thus needs to be incorporated as a local contribution $k\hat{u}u$.

Ritz-Galerkin methods seek approximations of the solutions within user defined subspaces. These are typically built using polynomial or trigonometric functions of space. Most finite element solutions are thus built using piece-wise polynomials on a decomposition of space in a set of simple geometrical supports (or elements).

Taking again our beam example, one can seek a solution within the subspace of second order polynomials $u(x, t) = q_0 1 + q_1 x + q_2 x^2$. 1, x , and x^2 are basis vectors (called T_i elsewhere in these notes). $\{q_i\}$ are called degree of freedom (DOF). For a beam clamped at both ends, the only cinematically admissible polynomial is $x(x - L)$, one thus seeks a solution of the form $u(x, t) = (x(x - L))q(t)$.

A Ritz approximation (also called assumed modes method) is obtained by assuming $u(x, t)$ and $\hat{u}(x)$ to be within the subspace generated by a set of functions $T_i(x)$, that is $u(x, t) = \sum_i T_i(x)q_i(t)$. Since u and \hat{u} appear linearly in the strain and kinetic energies one has

$$\{\hat{q}\}^T [K] \{q\} = \int_0^L \frac{\partial \hat{u}}{\partial x} EA \frac{\partial u}{\partial x} \quad \text{thus} \quad K_{ij} = \int_0^L \frac{\partial T_i}{\partial x} EA \frac{\partial T_j}{\partial x} \quad (3.11)$$

Similarly

$$\{\hat{q}\}^T [M] \{\ddot{q}\} = \int_0^L \hat{u} \rho A \frac{\partial^2 u}{\partial t^2} \quad \text{thus} \quad M_{ij} = \int_0^L T_i \rho A T_j \quad (3.12)$$

Finally

$$\{\hat{q}\}^T \{b\} = \int_0^L \hat{u} f_v(x) \quad \text{thus} \quad b_i = \int_0^L T_i f_v(x) \quad (3.13)$$

The input matrix b corresponds to the force associated to a unit load. One assumes distributed loads to be of the form $f_v(x, t) = \sum f_{vk}(x)u_k(t)$, where f_{vk} are unit loads associated with inputs u_k . One thus has

$$b_{ik} = \int_0^L T_i f_{vk}(x) \quad (3.14)$$

The observation matrix c is simply derived from the kinematic assumption $u(x, t) = \sum_i T_i(x) q_i(t)$. Thus coefficients of the observation matrix for a displacement at x_0 is given by

$$c_i = T_i(x_0) \quad (3.15)$$

and for a strain ϵ_{xx}

$$c_i = \frac{\partial T_i(x_0)}{\partial x} \quad (3.16)$$

3.1.3 Time response : exact transition matrix and iterative methods

The resolution of linear differential equations is a classical problem treated in many textbooks (see a course on control theory [17] or vibration [16]). This section insists on two aspects : writing of the exact solution and basic recursive numerical approximations.

To compute the forced response of a set of linear differential equations, one first starts by determining particular solutions of the homogeneous equations (in the absence of external loads)

$$\{\dot{x}(t)\}_{N \times 1} = [A]_{N \times N} \{x(t)\}_{N \times 1} \quad (3.17)$$

For a scalar equation, the solutions are known to be of the form $\{x(t)\} = \{x(t_0)\} e^{\lambda(t-t_0)}$. For a matrix equation, one can write formally

$$x(t) = e^{[A](t-t_0)} x(t_0) \quad (3.18)$$

The definition of the matrix exponential is given formally by its series expansion

$$e^{[A](t-t_0)} = I + \frac{[A]t}{1!} + \frac{([A]t)^2}{2!} + \dots \quad (3.19)$$

Note : the computation of the matrix exponential is actually trivial if the matrix is diagonal (one has the scalar exponential of each term on the diagonal). The computation is thus performed using a state-transformation to the diagonal form. In the time domain the states of interest are particular solutions (modes) that verify $\{x(t)\} = \{x(0)\} e^{\lambda t}$.

One now obtains a general expression for the forced response. By deriving $e^{-At}x(t)$, one obtains from (3.18)

$$\frac{d}{dt} [e^{-At}x] = e^{-At} [\dot{x} - Ax] = e^{-At} Bu \quad (3.20)$$

which can be integrated, multiplied by e^{At} to give an exact expression of the forced response

$$x(t) = e^{[A](t-t_0)} x(t_0) + \int_{t_0}^t e^{[A](t-\tau)} [B] \{u(\tau)\} d\tau \quad (3.21)$$

In practical applications, rather than computing the transition matrix, one often prefers to use numerical integration procedures which give approximations of the model state x as a

function of time. This idea of state-transformations is however often used to limit the amount of computations needed for real time integration of models in specialized DSP (Digital Signal Processor).

The fundamental principle of integration schemes is to evaluate the time derivative (in the evolution equation (3.1)) and estimate change in state over a given time step. The explicit Euler algorithm thus uses

$$\{x(t^{k+1})\} = \{x(t^k)\} + \delta t \{\dot{x}(t^k)\} \quad (3.22)$$

It is called an explicit algorithm because the estimate of $x(t^{k+1})$ only depends on values at time t^k . Combining equation (3.22) and the evolution equation (3.1), leads to

$$\{x(t^{k+1})\} = [[I] + \delta t [A]] \{x(t^k)\} + \delta t [B] \{u(t^k)\} \quad (3.23)$$

which for a fixed time step is a simple recursive equation associated with a *discrete time* state space model.

Similarly the implicit Euler algorithm uses

$$\{x(t^{k+1})\} = \{x(t^k)\} + \delta t \{\dot{x}(t^{k+1})\} \quad (3.24)$$

It is called an implicit algorithm because the estimate of $x(t^{k+1})$ also depends on values at time t^{k+1} . Combining equation (3.24) and the evolution equation (3.1), leads to

$$\{x(t^{k+1})\} = [[I] - \delta t [A]]^{-1} \{x(t^k)\} + [[I] - \delta t [A]]^{-1} \delta t [B] \{u(t^{k+1})\} \quad (3.25)$$

Here discuss stability using a single degree of freedom model.

Precision will be discussed later for mechanical systems.

3.1.4 Frequency response

As for the case of the 1 DOF oscillator, one is often interested by the frequency response (in the Laplace or Fourier domains, [18, 17]). The Laplace transform of the time domain state space model (3.1) is thus given by

$$\begin{aligned} s \{x(s)\} - \{x(t_0)\} &= [A] \{x(s)\} + [B] \{u(s)\} \\ \{y(s)\} &= [C] \{x(s)\} + [D] \{u(s)\} \end{aligned} \quad (3.26)$$

leading to the input/output relation

$$\begin{aligned} \{x(s)\} &= [sI - A]^{-1} \{x(t_0)\} + [sI - A]^{-1} [B] \{u(s)\} \\ \{y(s)\} &= [C] [sI - A]^{-1} \{x(t_0)\} + [C] [sI - A]^{-1} [B] \{u(s)\} + [D] \{u(s)\} \end{aligned} \quad (3.27)$$

For a zero initial condition, this response is characterized by the transfer function

$$[H(s)] = [C] (s [I] - [A])^{-1} [B] + [D] \quad (3.28)$$

relating inputs and outputs by

$$\{y(s)\} = [H(s)] \{u(s)\} \quad (3.29)$$

The computation or experimental measurement of the transfer matrix $H(s)$ is one of the central problems of **modal analysis**. The inverse Laplace transform of H , call *impulse response* (or Green function), corresponds to a restriction for specific inputs /outputs of the transfer matrix considered in (3.21). One of the experimental techniques to measure transfer functions is to use an impact on the structure (typically with an instrumented hammer) and to measure the impulse response and estimate the transfer using the Fourier transform (more details in section 4.3).

For mechanical systems, the evolution (3.2) and observation (3.4) equations can be written in the frequency domain. In the Laplace domain, the input/output relation for zero initial conditions given by the following equations

$$\begin{aligned} [Ms^2 + Cs + K] \{q(s)\} &= [b] \{u(s)\} \\ \{y(s)\} &= [c] \{q(s)\} \end{aligned} \quad (3.30)$$

from which one obtains the expression of transfer functions

$$\{y(s)\} = [H(s)] \{u(s)\} = [c] [Ms^2 + Cs + K]^{-1} [b] \{u(s)\} \quad (3.31)$$

3.2 Modal coordinates for elastic models

In mechanical applications, one typically computes elastic modes and generates modal models. Since this is the most common usage, it is described here while section 3.3 addresses the same transformation for state space models.

3.2.1 Computation of normal modes

Modes, in general, are shapes fixed in space that have an harmonic response in time $\{q(t)\} = \{\phi_j(x)\} e^{\lambda_j t}$ (possibly with a complex pole if damping is present). Physically, this space time decomposition is important because it will propagate in time with the same shape if excited by any initial condition. But let us start with the numerical aspect.

The computation of frequency and time responses of dynamics systems is greatly facilitated by the use of **spectral decompositions**. For structures, **damping** is often neglected at first. For good reasons (it is often small, it can be reintroduced later with small errors) and a bad one (one rarely has enough knowledge of the physical parameters that drive dissipation). The first step of most analyses is thus to only consider mass and stiffness properties (assume an **elastic** behavior). Even when damping modeling is added [1], this step remains crucial.

Modes associated with an elastic model, called **normal modes**, are solutions of the eigenvalue problem

$$- [M] \{\phi_j\} \omega_j^2 + [K]_{N \times N} \{\phi_j\}_{N \times 1} = \{0\}_{N \times 1} \quad (3.32)$$

Being the result of an approximation of the kinetic and strain energies, the mass M matrix is symmetric positive and the stiffness K positive semi-definite (vectors of its null space are rigid body modes). A well known linear algebra result is the existence of N distinct eigenvectors (where N is the number of degrees of freedom).

For distinct eigenvalues, one can show that eigenvectors are orthogonal with respect to the mass matrix as follows. By left multiplying (3.32) by ϕ_k for two distinct values of j and k , one obtains

$$\begin{aligned} -\{\phi_k\}^T [M] \{\phi_j\} \omega_j^2 + \{\phi_k\}^T [K] \{\phi_j\} &= 0 \\ -\{\phi_j\}^T [M] \{\phi_k\} \omega_k^2 + \{\phi_j\}^T [K] \{\phi_k\} &= 0 \end{aligned} \quad (3.33)$$

One then takes the difference of those two equations and using the symmetry of the matrices one has

$$(\omega_j^2 - \omega_k^2) \{\phi_k\}^T [M] \{\phi_j\} = 0 \quad (3.34)$$

Thus if the frequencies are distinct

$$\{\phi_k\}^T [M] \{\phi_j\} = 0 \quad , \quad \forall j \neq k \quad (3.35)$$

For multiple eigenvalues, one can also show that there exists a possible choice of eigenvectors such that all eigenvectors are orthogonal. The demonstration can be found in [16], but the only the result and the procedure (a simple Gramm-Schmitt Orthogonalization for example) is important.

The conditions thus obtained are called **orthogonality conditions** with respect to the mass are often used in matrix form

$$[\phi]^T [M] [\phi] = \begin{bmatrix} \mu_j \end{bmatrix} \quad (3.36)$$

where the scalar μ_j , called **generalized masses**, correspond to the kinetic energy norm of eigenvector ϕ_j . They thus depend on how the vector is normalized and are thus arbitrary. Typical choices for this normalization are

- mass normalize eigenvectors ϕ_j , that is $\mu_j = 1$;
- stiffness normalize eigenvectors ϕ_j , that is $\mu_j \omega_j^2 = 1$ (note this normalization makes no sense for rigid body modes) ;
- set on particular component to 1. One typically considers the maximum component (or the maximum translation component). For $\tilde{\phi}_j$ a vector normalized such that $[c] \{\tilde{\phi}_j\} = 1$, one has $\{\tilde{\phi}_j\} = \{\phi_j\} / ([c] \{\phi_j\})$ et $\mu_j = ([c] \{\phi_j\})^{-2}$
- normalize the vector to one using the Euclidian norm. While sometime encountered (mostly in experimental work), this approach only makes physical sense if all DOFs have the same nature.

In the following sections, one will assume $\mu_j = 1$, for the simple reason that it simplifies most equation writing.

In class, one shows pictures to illustrate the concept and illustrates the convergence of the generalized mass to the true mass.

One also shows that there exists an **orthogonality condition** with respect to the stiffness matrix

$$[\phi]^T [K] [\phi] = \begin{bmatrix} \mu_j \omega_j^2 \end{bmatrix} \quad (3.37)$$

Orthogonality conditions are quite essential. In particular they can be used to express the inverses of the mass and stiffness (in cases without rigid body modes) matrices

$$[M]^{-1} = [\phi] \begin{bmatrix} \mu_j^{-1} \end{bmatrix} [\phi]^T \quad [K]^{-1} = [\phi] \begin{bmatrix} (\mu_j \omega_j^2)^{-1} \end{bmatrix} [\phi]^T \quad (3.38)$$

One also has a direct expression of the dynamic flexibility matrix

$$([M]s^2 + [K])^{-1} = [\phi] \left[\frac{1}{\mu_j(s^2 + \omega_j^2)} \right] [\phi]^T = \sum_{j=1}^N \frac{\{\phi_j\} \{\phi_j\}^T}{\mu_j (s^2 + 2\zeta_j \omega_j s + \omega_j^2)} \quad (3.39)$$

which is used to compute transfer functions (3.31) as a series of modal contributions. In this expression **modal damping ratio** ζ_j is introduced in most applications. This ratio is either derived from tests or taken as a design parameter. $\zeta_j \in [10^{-4} \ 5.10^{-3}]$ is typical for pure metals. For assembled structures one often takes $\zeta_j \in [5.10^{-3} \ 0.02]$.

3.2.2 Modes of continuous systems

Modes of continuous systems are computed for a few classical configurations. It is useful to know the procedure for the linear PDE found for the beam and 2D plate equations. Other configurations can be found in books.

Modes are harmonic solutions of the dynamic equations in the absence of external loading. *Harmonic* means that the time dependence is of the form $e^{\lambda_j t}$. One thus seeks solutions of the form $u(x, t) = \phi(x)e^{\lambda_j t}$. In the case of undamped systems, the poles are purely imaginary $\lambda_j = i\omega_j$.

In class do the case of traction/compression, beam bending, plate bending.

3.2.3 Time domain modal model

The model having N distinct eigenvectors, these form a full basis of the vector space spanned by the DOFs. One can thus always write the equations of motion using principal coordinates defined by

$$\{q\} = [\phi] \{p\} \quad (3.40)$$

By placing q in the equations of motion one obtains

$$[M] [\phi] \{\ddot{p}\} + [K] [\phi] \{p\} = F(t) \quad (3.41)$$

which one then left multiplies by $[\phi]^T$ (this is the principle of virtual work). Then using the mode orthogonality conditions, one obtains new equations of motion in **principal coordinates**

$$[I] \{\ddot{p}\} + [\phi^T C \phi] \{\dot{p}\} + [\Omega^2] \{p\} = [\phi^T b] \{u(t)\} \quad (3.42)$$

The **modal damping** (*amortissement modal ou hypothèse de Basile* assumption assumes that $[\phi^T C \phi] \approx [2\zeta_j \omega_j]$

The mass $[I]$ and stiffness $[\Omega^2]$ matrices are now diagonal. One can thus solve each scalar equation (row) in (3.42) separately. Thus for initial conditions $\{q(0)\}, \{\dot{q}(0)\}$, the initial modal states are given by $\{p(0)\} = [\phi]^T [M] \{q(0)\}, \{\dot{p}(0)\} = [\phi]^T [M] \{\dot{q}(0)\}$ and the free response of each mode is that of a 1DOF oscillator

$$\{p_j(t)\} = \{\phi_j\}^T \{q(0)\} \cos(\omega_j t) + \{\phi_j\}^T [M] \{\dot{q}(0)\} \sin(\omega_j t) \quad (3.43)$$

The global response, can then be computed summing the series of modal contributions

$$\{q(t)\} = \sum_{j=1}^N \{\phi_j\} \{p_j(t)\} = \sum_{j=1}^N \{\phi_j\} \{\phi_j\}^T (\{q(0)\} \cos(\omega_j t) + \{\dot{q}(0)\} \sin(\omega_j t)) \quad (3.44)$$

3.2.4 Frequency response

Frequency responses are computed for particular loads, at particular response points (sensors) and for a given frequency range. Taking principal coordinates $\{q\} = [\phi] \{p\}$, and using the orthogonality condition(3.39), one can rewrite equation (3.31) as a series of modal contributions

$$H(\omega) = [c] [-M\omega^2 + Ci\omega + K]^{-1} [b] \approx \sum_{j=1}^N \frac{[c] \{\phi_j\} \{\phi_j\}^T [b]}{-\omega^2 + 2\zeta_j \omega_j i\omega + \omega_j^2} \quad (3.45)$$

An exact approximation would consider all modes, but as for most series a truncation is performed. The truncation is based on frequency band of interest (keeping NM modes who have their resonance within the band). The truncated part should be approximated using so called **static correction** (or **residual vectors**) that will be detailed in section 5.3.1.

It is quite important to understand the relation between the response $\{y\} = [H] \{u\}$ and the modeshapes $\{\phi_j\}$. Figure 3.2 illustrates how the imaginary part of isolated resonances directly corresponds to modeshapes. For a cantilever beam at resonance, the imaginary part of the transfer is given by $\frac{[c] \{\phi_j\} \{\phi_j\}^T [b]}{2i\zeta_j \omega_j^2}$. For a fixed excitation, $\{\phi_j\}^T [b]$ is constant and one can directly read the relative amplitudes at sensors $c \{\phi_j\}$ by looking at the various transfers (the blue lines correspond to modeshapes).

To understand this one need to decompose a modal contribution as the product of

- $[c] \{\phi_j\}$ the modal observability
- $\{\phi_j\}^T [b]$ the modal controlability
- $\frac{1}{s^2 + 2\zeta_j \omega_j s + \omega_j^2}$ the modal amplification.

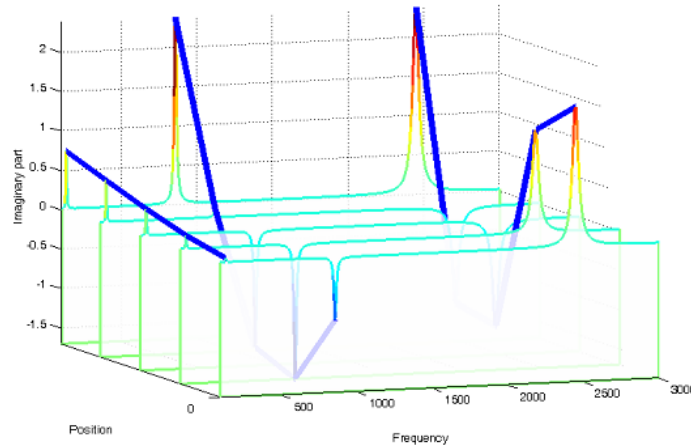
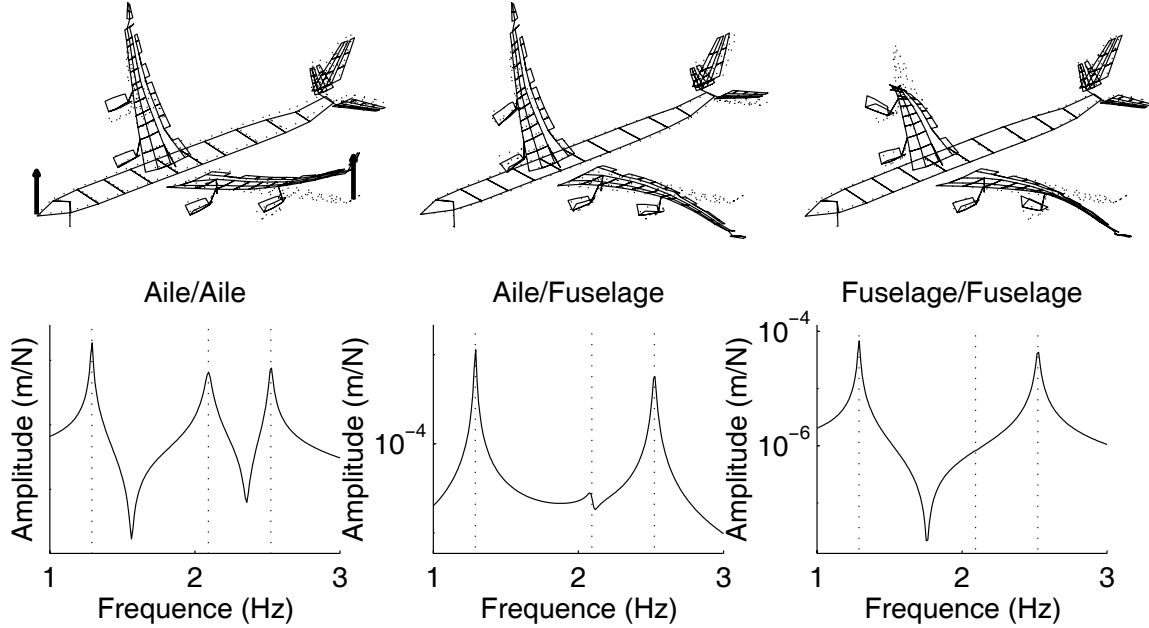


FIGURE 3.2 – Imaginary part transfer function as a function of position on a cantilever beam

The contribution levels of each mode are often quite different. To understand the parameters influencing the contribution of a given mode, let us consider 3 modes of the Airbus A340 shown below. One is interested in 3 transfers corresponding to vertical excitation and sensors placed at the wing tips and the front of the fuselage.

For the wing/wing transfer, the modal observabilities $c\phi_j$ and controllabilities $\phi_j^T b$ of the 3 modes are similar. For similar damping levels the transfer has 3 peaks of similar levels. The

second mode (3 node bending) is anti-symmetric and thus has very low vertical motion of the fuselage. For the cross transfer wing/fuselage, the modal contribution driven by the product $c\phi_j\phi_j^T b$, is much lower for the second mode than the others, the associated peaks shows but much less. For the fuselage/fuselage transfer both the observability and controllability are small, the second mode thus has a negligible contribution and now longer shows in the transfer.



It is important to understand that in real life transfers, not all modes are visible and the level of each contribution depends on three factors : observability, controllability, and damping level.

Associated subjects : Reciprocity and collocated inputs/outputs. Convergence of the modal series for collocated inputs.

3.3 Modes in a more general setting

3.3.1 State-space model modes

In control theory, one often considers the free response of state space models. If can be shown that this simply related to a so called *state transition matrix* $\Phi(t) = e^{[A]t}$. The standard procedure to evaluate this matrix numerically is to use a spectral decomposition of the $[A]$ matrix. Assuming that $[A]$ can be diagonalized, one has two bases of left $[\theta_L]$ and right $[\theta_R]$ eigenvectors solutions of the problems

$$[A] \{\theta_{jR}\} = \lambda_j \{\theta_{jR}\} \quad \text{and} \quad \{\theta_{jL}\}^T [A] = \{\theta_{jL}\}^T \lambda_j \tag{3.46}$$

and such that

$$[\theta_L]^T [A] [\theta_R] = [\Lambda] \quad \text{et} \quad [\theta_L]^T [\theta_R] = [I] \tag{3.47}$$

One then finds that $A^n = [\theta_R] [\Lambda]^n [\theta_L]^T$ and thus that

$$e^{[A]t} = [\theta_R] e^{([\Lambda]t)} [\theta_L]^T = \sum_{j=1}^N \{\theta_{jR}\} e^{(\lambda_j t)} \{\theta_{jL}\}^T \tag{3.48}$$

It thus appears that the state transition matrix can be written as a series of modal contributions. The result is true for almost every linear differential equation.

The time domain result on the transition matrix, can be translated into the frequency domain. The transfer matrix can be decomposed in a series of low order rational fractions associated with each mode

$$[H(s)] = \sum_{j=1}^N \frac{[C] \{\theta_{jR}\} \{\theta_{jL}\}^T [B]}{s - \lambda_j} + [D] \quad (3.49)$$

When the original state-space model is real, modes are either real or come in complex conjugate pairs (poles and modeshapes are conjugate). One can thus group modes by pair and build a real valued state-space model with a block diagonal form associated with each pair

$$\begin{aligned} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} &= \begin{bmatrix} [0] & [I] \\ -[\omega_j^2] & -[2\zeta_j \omega_j] \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} \{u(t)\} \\ \{y(t)\} &= [C_1 \ C_2] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \end{aligned} \quad (3.50)$$

where the blocks of matrices B_1 , B_2 , C_1 , C_2 are given by

$$\begin{aligned} \begin{Bmatrix} B_{j1} \\ B_{j2} \end{Bmatrix} &= 2 [\operatorname{Re}(\theta_{jL}^T B) \ \operatorname{Im}(\theta_{jL}^T B)] \begin{bmatrix} 1 & 0 \\ \zeta_j \omega_j & -\omega_j \sqrt{1 - \zeta_j^2} \end{bmatrix} \\ \begin{Bmatrix} C_{1j} \\ C_{2j} \end{Bmatrix} &= \frac{1}{\omega_j \sqrt{1 - \zeta_j^2}} \begin{bmatrix} \omega_j \sqrt{1 - \zeta_j^2} & \zeta_j \omega_j \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \operatorname{Re}(C \theta_{jR}) \\ \operatorname{Im}(C \theta_{jR}) \end{bmatrix} \end{aligned} \quad (3.51)$$

3.3.2 State/DOF transformations, model reduction

As shown in the block-diagram of figure 3.1, the system is characterized by the relation between inputs u and outputs y . Thus a bijective transformation of the state space $\{x\} = [T] \{\tilde{x}\}$ leads to another state space model of the *same* system

$$\begin{aligned} \{\dot{\tilde{x}}(t)\} &= [T^{-1}AT] \{\tilde{x}(t)\} + [T^{-1}B] \{u(t)\} \\ \{y(t)\} &= [CT] \{\tilde{x}(t)\} + [D] \{u(t)\} \end{aligned} \quad (3.52)$$

For these two models, one talks of different *realizations*. The construction of uniquely defined state space representations is a necessary condition for the term by term comparison of state space models.

Model reductions methods, discussed in chapter 5, are methods that introduce state transformations where a number of states can be eliminated (considered to be zero) while retaining a good approximation of the system dynamics (u/y relation).

For mechanical simulations, one does not use the full model (3.2) directly but rather an approximation within a subspace. One thus assumes that

$$\{q\}_{N \times 1} = [T]_{N \times NR} \{q_R\}_{NR \times 1} \quad (3.53)$$

and projects (3.2) on the dual subspace T^T , which leads to approximations of the transfer functions as

$$[H(\omega)] = [c] (-[M]\omega^2 + [K])^{-1} [b] \approx [cT] (-[T^TMT]\omega^2 + [T^TKT])^{-1} [T^Tb] \quad (3.54)$$

or simulations of the time domain response with

$$\begin{aligned} [T^TMT] \{\ddot{q}_R\} + [T^TCT] \{\dot{q}_R\} + [T^TKT] \{q_R\} &= [T^Tb] \{u(t)\} \\ \{y(t)\} &= [cT] \{q_R\} \end{aligned} \quad (3.55)$$

One will note that, for a non singular T ($\{q\} = [T] \{q_R\}$ bijective), the input/output relation between u and y is preserved. One says that has a different **realization** of the same system. One can say that transfer are **objective** quantities (physical quantities uniquely defined by the specification of the input and the measurements) while the DOFs q (or states X in state-space models) are only objective if they are also defined as outputs.

Model reduction methods (modal analysis, condensation, substructuring, superelements, domain decomposition methods, ..., see chapitre 5) seek to justify a priori (or through error estimators) the validity of a state transformation where some states are eliminated (T is rectangular, it spans a subspace of the initial vector space where q is defined).

The ability to correctly use a reduced basis depends on two main assumptions. One only cares about a limited frequency band and one makes prior assumptions on the spatial distribution of applied loads.

3.3.3 Typical model limitations

State space models are always approximations of the dynamics. A few of their classical limitations are

- true systems are rarely completely linear. They are often also somewhat time varying. Model parameters are always approximations.
- model representativity is always over a limited frequency band. This typically corresponds to the modal truncation discussed in section 5.3.1. It is seen there that the effect of high frequency modes can/should be approximated by a constant feed-through which corresponds to the $[D] \{u\}$ term of the standard state space model. One will however note that for velocity a $[D] \{\dot{u}\}$ should be used, when it does not exist in standard state space representation. One must thus introduce additional states for the correction.
- when using real actuators there often is a significant distinction between the desired input and the effective one. The figure below illustrates the non perfect transfer of an amplifier used to control piezoelectric patches.

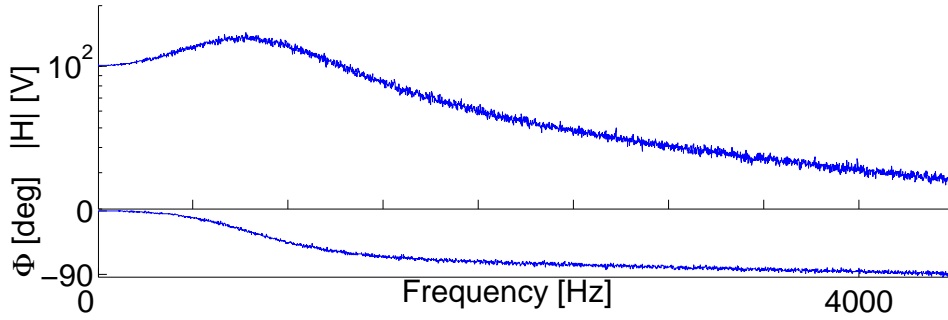


FIGURE 3.3 – Transfer of an amplifier for piezoelectric patches

- digital systems have time delays which can often induce significant phase shifts. The Laplace transform of a time delay of length τ is given by $e^{-s\tau}$. At low frequencies, one generally approximates time delays by a Pade approximation [17], such as the second order approximation

$$e^{-s\tau} \approx \frac{1 - (\tau s/2) + (\tau^2 s^2/12)}{1 + (\tau s/2) + (\tau^2 s^2/12)} \quad (3.56)$$

Methods that account for the difference between model and reality are called **robust**.

3.4 Damping models

3.4.1 Modal damping

The most common approach is the modal damping assumption. Elastic modes are computed first and a damping ratio ζ_j is given for each mode. The values chosen can be derived from test or chosen as a design objective (usually based on a number of earlier experiments). In modal coordinates, the viscous damping matrix is given by

$$\Gamma = [\Phi]^T [C] [\Phi] = \left[\sqrt{2\zeta_j \omega_j} \right] \quad (3.57)$$

Considering the fact that modes are mass orthonormal ($[\Phi]^T M [\Phi] = [I]$) clearly shows that $[\Phi]^{-1} = ([\Phi]^T M)$ as a result a direct expression of modal damping C is

$$C = [M\phi] \left[\sqrt{2\zeta_j \omega_j} \right] [\phi^T M] \quad (3.58)$$

[19] illustrates how this formula can be used in practice.

3.4.2 Rayleigh or proportional damping

Historically, modal damping was associated to the **proportional damping model** introduced by Lord Rayleigh which assumes the usefulness of a global viscously damped model with a dynamic stiffness of the form

$$[Z(s)] = [Ms^2 + (\alpha M + \beta K)s + K]$$

Caughey [20] also introduced generalizations with $M^a \times K^b$ and some software allow the use of different α, β coefficients for each material. Illustrations can be found in [19].

While this model indeed leads to a modally damped normal mode model, the α and β coefficients can only be adjusted to represent physical damping mechanisms over very narrow frequency bands. The modal damping matrix thus obtained writes

$$\Gamma = \left[\alpha + \beta \omega_j^2 \right]$$

which leads to damping ratios

$$2\zeta_j = \frac{\alpha}{\omega_j} + \beta \omega_j$$

3.4.3 More physical damping models : viscoelasticity, hysteresis

For more realistic approaches, you should look into the definition of *hysteretic* or **structural damping**, the notion of loss factor, ... details can be found in Ref [1] (available at <https://www.sdtools.com/pdf/visc.pdf>, Code_Aster www.code-aster.org/doc/man_r/r5.05.04.pdf, the Wikipedia page on *Viscoelasticity*).

3.5 Seismic response

3.5.1 Case of a rigid base

One considers a model with a rigid base. Base motion is described by at most 6 DOFs q_I while motion of the rest of the structure is described by q_C . The frequency domain equation of motion becomes

$$\begin{bmatrix} Z_{II}(s) & Z_{IC}(s) \\ Z_{CI}(s) & Z_{CC}(s) \end{bmatrix} \begin{Bmatrix} \langle q_I(s) \rangle \\ \langle q_C(s) \rangle \end{Bmatrix} = \begin{Bmatrix} R_I(s) \\ \langle F_C \rangle \end{Bmatrix} \quad (3.59)$$

where $\langle \rangle$ is used to indicate given quantities. The transfer functions characterizing the unknown quantities are defined by

$$\begin{Bmatrix} R_I \\ q_C \end{Bmatrix} = \begin{bmatrix} s^2 M_{II} & -T_{IC} \\ T_{CI} & H_{CC} \end{bmatrix} \begin{Bmatrix} q_I \\ F_C \end{Bmatrix} = \begin{bmatrix} Z_{II} - Z_{IC} Z_{CC}^{-1} Z_{CI} & Z_{IC} Z_{CC}^{-1} \\ -Z_{CC}^{-1} Z_{CI} & Z_{CC}^{-1} \end{bmatrix} \begin{Bmatrix} q_I \\ F_C \end{Bmatrix} \quad (3.60)$$

This equation clearly shows that the singularities in M_{II} , T_{IC} and H_{CC} are associated with the modes of Z_{CC}^{-1} where the interface is fixed.

Given mass normalized fixed interface modes ϕ_{jC}, ω_j solutions of $Z_{CC}(\omega_j) \{\phi_{jC}\}$, an input shape matrix b characterizing the spatial distribution of $F_C = [b] \{u\}$, an output shape matrix c describing measurements $\{y\} = [c_I \ c_C] \{q\}$, rigid body modes

$$\phi_r = \begin{bmatrix} \phi_{rI} \\ \phi_{rC} \end{bmatrix} = \begin{bmatrix} I \\ -K_{CC}^{-1} K_{CI} \end{bmatrix} \quad (3.61)$$

and assuming modal damping for the fixed interface modes, one can establish [21] the expressions for the *dynamic flexibility*

$$H_{CC}(s) = \sum_{j=1}^{NC} \frac{\{c_C \phi_{jC}\} \{\phi_{jC}^T b\}}{s^2 + 2\zeta_j \omega_{jC} s + \omega_{jC}^2} \quad (3.62)$$

transmissibility

$$T_{CI}(s) = \sum_{j=1}^{NC} \{c_C \phi_{jC}\} \{\phi_{jC}^T b_R\} \frac{\zeta_j \omega_{jC} s + \omega_{jC}^2}{s^2 + 2\zeta_j \omega_{jC} s + \omega_{jC}^2} - M_{CC}^{-1} M_{CI} \quad (3.63)$$

and *dynamic mass*

$$M_{II}(s) = \sum_{j=1}^{NC} \{b_R^T \phi_{jC}\} \{\phi_{jC}^T b_R\} \frac{\zeta_j \omega_{jC} s + \omega_{jC}^2}{s^2 + 2\zeta_j \omega_{jC} s + \omega_{jC}^2} + M_{II} - M_{IC} M_{CC}^{-1} M_{CI} \quad (3.64)$$

with

$$b_R = [M_{CI} \ M_{CC}] \begin{bmatrix} I \\ -K_{CC}^{-1} K_{CI} \end{bmatrix} = M_{CI} - M_{CC} K_{CC}^{-1} K_{CI} \quad (3.65)$$

It clearly appears that b_R plays the role of an input shape matrix so that $\phi_{jC}^T b_R$ is called a modal participation factor with the same characteristics as the modal controllability describe for load inputs.

The mass terms in the transmissibility and dynamic masses are due to the mass contribution of DOFs linked to the base M_{CI} and M_{II} . Thus unless the finite elements connected to the base have a very large mass, these terms are negligible.

3.5.2 Effective masses and stiffness

The concept of effective mass is used to estimate whether the contribution of a given mode is important in the seismic response. One can easily show that the dynamic mass converges at $s = 0$ to the rigid body mass of the system \bar{M}_{II}

$$\bar{M}_{II} = \phi_r^T M \phi_r = M_{II} - M_{IC} K_{CC}^{-1} K_{CI} - K_{IC} K_{CC}^{-1} M_{CI} + K_{IC} K_{CC}^{-1} M_{CC} K_{CC}^{-1} K_{CI} \quad (3.66)$$

Thus

$$\sum_{j=1}^{NC} \{b_{Rk}^T \phi_{jC}\} \{\phi_{jC}^T b_{Rk}\} = \bar{M}_{II} - (M_{II} - M_{IC} M_{CC}^{-1} M_{CI}) \approx \bar{M}_{II} \quad (3.67)$$

for an excitation in a given direction k one can thus define the *effective mass* of fixed interface mode j by

$$m_{jk} = \frac{\{b_{Rk}^T \phi_{jC}\} \{\phi_{jC}^T b_{Rk}\}}{(\bar{M}_{II})_{kk}} \quad (3.68)$$

and see that the sum of the effective masses converges to 1. It is thus quite usual to only consider modes whose effective mass is a significant fraction of unity, or to truncate when 90% of the mass is represented.

The corresponding effective stiffness concept is linked to the convergence of H towards the static flexibility

$$\sum_{j=1}^{NC} k_{jk} = \sum_{j=1}^{NC} \frac{\{b_k^T \phi_{jC}\} \{\phi_{jC}^T b_k\}}{b_k K_{CC}^{-1} b_k} = 1 \quad (3.69)$$

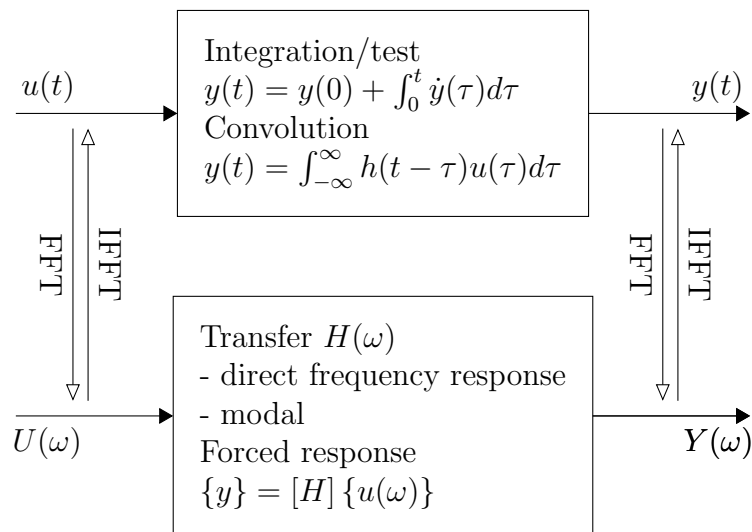
Effective stiffness are however rarely used to motivate modal basis truncation.

Note that the effective mass and stiffness concepts are intimately linked to the modal controllability concept but provide the additional information about the fraction of the static response contributed by each mode.

Chapitre 4

Time and frequency domains, signal processing basics

Most of the processes addressed in this course are linked to transformations between the frequency and time domains. In simulation, direct time integration is often replaced by transform to frequency domain, product to obtain forced response and back transform to obtain the transient response. In test the transfer functions are estimated from test, ... It is thus critical to have in mind the main properties and pathologies of Fourier transform.



Many physical phenomena are studied by building a model from specific experiments. The model is not obtained from physical knowledge, but inferred from the experiment. This analysis is mostly based on digital signal processing and Fourier analysis. Many references exist in this area [22, 23, 24], this chapter is thus limited to key notions. It is significantly inspired from [25] which summarizes classical theory. *In class, a complement on modal analysis hardware (shakers, sensors, ...) is given.*

The model characterization being typically done in the Fourier domain, the main properties of the transform are given in 4.1. Fundamental difficulties associated with sampling and limited time measurements are shown in section 4.2. Finally section 4.3 discusses transfer function estimation based on time measurements (this is called *non-parametric identification*).

4.1 A few things about the Fourier Transform

4.1.1 Continuous and discrete transforms

J.B. Fourier demonstrated during the 18th century, that a periodic function over interval $[0 T]$ can be represented as the transform pair

$$Y(k\Delta f) = \frac{1}{T} \int_0^T y(t)e^{-j2\pi\Delta f t} dt \quad \text{with} \quad y(t) = \sum_{k=-\infty}^{+\infty} Y(k\Delta f)e^{j2\pi k\Delta f t} \quad (4.1)$$

where $T = 1/\Delta f$ (frequency given in *Hz*, one also uses *rad/s*). The Fourier spectrum $Y(k\Delta f)$ is generally complex. It is discrete for periodic signals.

Non periodic signals are obtained when the period T tends to infinity, one then uses the continuous transform

$$Y(\omega) = \int_{-\infty}^{+\infty} y(t)e^{-j\omega t} dt \quad \text{and} \quad y(t) = \int_{-\infty}^{+\infty} Y(\omega)e^{j\omega t} d\omega \quad (4.2)$$

where $Y(\omega) = Y(2\pi f)$ is called the direct Fourier transform and $y(t)$ the inverse transform. For non-periodic signals, the spectrum is continuous and generally complex.

In general, signals are sampled. One thus represents the continuous function $y(t)$ by a series in a discrete number of time samples that are **evenly** distributed. One then considers the Fourier transform pair

$$Y(f) = \sum_{n=-\infty}^{+\infty} y(n\Delta t)e^{-j\omega n\Delta t} \quad \text{and} \quad y(n\Delta t) = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} Y(\omega)e^{j\omega n\Delta t} d\omega \quad (4.3)$$

where n is a counter for time samples and $\omega_s = 2\pi/\Delta t$ is the sampling frequency. The Fourier transform of a sampled signal leads to a periodic function in the frequency domain.

In practice, one only measures a finite number of samples and generates a discrete Fourier transform (DFT) defined as

$$Y(k\Delta f) = \frac{1}{N} \sum_{n=0}^{N-1} y(n\Delta t)e^{j2\pi nk/N} \quad \text{and} \quad y(n\Delta t) = \sum_{k=0}^{N-1} Y(k\Delta f)e^{j2\pi nk/N} \quad (4.4)$$

with N samples, over times $t_k = [0\dots k\dots N-1]\Delta t$ and frequencies

$$f_k = \frac{[0\dots k\dots N-1]}{N\Delta t} = [0\dots k\dots N-1] \Delta f \quad (4.5)$$

with $\Delta f = 1/T$.

A few essential properties are listed below. You should read them and make sure that they make sense to you. If not, you should find a textbook on Fourier transforms.

- $Y(0)$ (and $Y(N/2 - 1)$ if N is even) are real valued for a real y . Points symmetric with respect to $(N - 1)/2$ (for N odd) or $N/2 - 1$ (for N even) are conjugate.

- Direct evaluations of DFT require N^2 operations. The FFT algorithm introduced a method requiring $N \log N$ operations for sample length in powers of 2. Modern libraries (FFTW in particular) achieve similar performance for arbitrary values of N . Unless you are using specific DSP hardware, which sometimes include FFT logic, there is thus no restriction on N .
- The main parameters of a measurement are the period $T = N\Delta t = 1/\Delta f$ (measurement length) which is directly related to the frequency sampling $\Delta f = 1/T$, the (time) sampling frequency $f_s = N/T$ related to the time sampling $\Delta t = 1/f_s$, the maximum frequency of interest in the spectrum $f_{max} \leq f_s/2$ (Shannon's theorem).
- For transient signals that one may want to pad with zeros, one prefers to use a scaled version of the FFT that is invariant when adding more zeros

$$Y(k\Delta f) = \frac{T}{N} \sum_{n=0}^{N-1} y(n\Delta t) e^{j2\pi nk/N} \quad \text{and} \quad y(n\Delta t) = \frac{N}{T} \sum_{k=0}^{N-1} Y(k\Delta f) e^{j2\pi nk/N} \quad (4.6)$$

- The Fourier transform is linear. If $Y_i(f)$ is the transform of $y_i(t)$, then $Y_1+Y_2 = \mathcal{F}(y_1+y_2)$ and $\alpha Y_1 = \mathcal{F}(\alpha y_1)$.
- $\mathcal{F}(y(at)) = Y(f/a)/|a|$. This inverse relation between time and frequency tells us that signals that change fast contain high frequencies.
- $\mathcal{F}(y(t-t_0)) = Y(f)e^{-j2\pi ft_0}$. One thus sees that a time delay only affects the phase of the Fourier transform (shift of $-j2\pi ft_0$).
- The energy of the signal can be computed in both the time and frequency domain (Parseval's theorem)

$$E = \int_{-\infty}^{+\infty} |g(t)|^2 dt = \int_{-\infty}^{+\infty} |G(f)|^2 df \quad (4.7)$$

where $|G(f)|^2$ is called the power spectral density and is the Fourier transform of the auto-correlation function.

For continuous spectra, frequency components have the dimension of a spectral density (hence the name). They must be integrated over a finite width to lead to a finite energy. For discrete spectra, each component has dimension of an energy. The use of *spectral density* is thus a misnomer.

- To interpolate a signal a very good approach is to pad its spectrum with zeros. To increase a frequency resolution you must use a longer time signal (or pad it by zeroes which is OK for transients).
- The Fourier transform of the time integration of $g(t)$ corresponds to a division by $i2\pi f$ of $G(f)$. The Fourier transform of $dg(t)/dt$ is given by $i2\pi fG(f)$.
- The transform of the product of two functions $a(t)$ and $b(t)$ is given by

$$c(t) = \int_{-\infty}^{+\infty} a(\tau)b(t-\tau)d\tau = a \star b \quad (4.8)$$

This product will be used in the next section to understand the effect of sampled measurements on the estimate of continuous transforms. These effects are known as leakage and *windows* are used to control it.

One also shows that the Fourier transform of a convolution product is the product of the Fourier transforms $x(t) = h(t) \star f(t) \Rightarrow X(f) = H(f)F(f)$. This property is a key motivation for the use of frequency domain transfers in structural dynamics.

4.1.2 Spectrum shift and zoom FFT

The inverse Fourier transform of $Y(f - f_0)$ is $y(t)e^{j2\pi f_0 t}$. This relation is the basis for Fourier transforms with selection of a base frequency band that does not start at zero. This is used to obtain good frequency resolutions even at high frequencies without storing very long measurements spans.

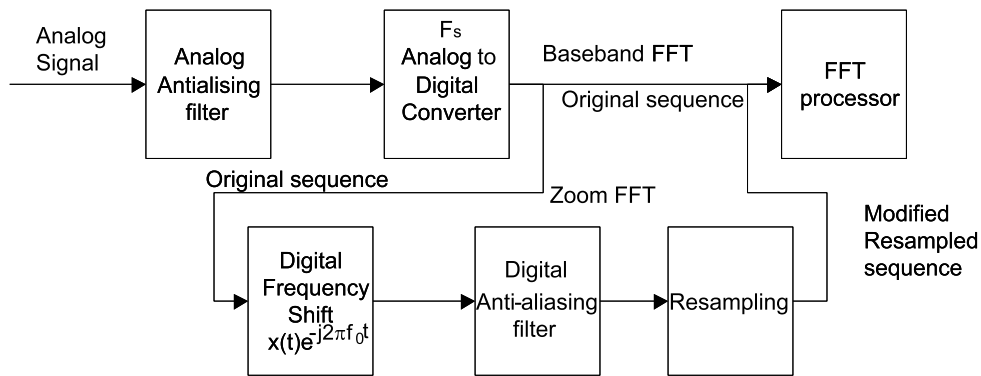


FIGURE 4.1 – Principle of the real time zoom

The most common technique, exposed here, is the real time zoom (for an implementation see [fe_curve](#)). Using the code below

```
Y=fft(y.*exp(-sqrt(-1)*2*pi*ftarg*t)); % shift and transform
N=length(t)/Nsub; Y(N:end-N)=0; % filter
Y(end:-1:end-N+1)=conj(Y(2:N+1)); % make symmetric
Y(1)=real(Y(1));
x=real(ifft(Y));
tx=t(1:Nsub:end); x=x(1:Nsub:end); % resample
fx=[1/diff(tx(1:2))*[0:length(tx)-1]'/length(tx)]+ftarg;
Y=fft(x)*Nsub;
```

For other methods see Ref.[24]. One starts by modulating the measured signal $y(t)e^{j2\pi f_0 t}$ which shifts the origin of frequencies towards f_0 .

One then need to resample. In [fe_curve](#) this is done exactly by computing the transform and performing a perfect band pass filtering before transforming back to the time domain. The really useful alternative is of course to perform the filtering and resampling using capabilities of a DSP (Digital Signal Processor) without performing a transform.

After resampling the modulated and filtered input, the shifted spectrum can be computed using the usual transform. The real time zoom does not change the fundamental fact that the frequency resolution is the inverse of the measurement length $\Delta F = 1/T$. It does however limit the amount of data that need to be taken out of the acquisition system.

4.2 Discrete transform errors

Many errors can occur in discrete measurements. Thus one will often have saturation, quantization errors, radioelectric and triboelectric alteration of the measured signal, ... The two most common effects are however due to the limited sampling frequency (which leads to aliasing (*repliment*) and the limited duration of measurements which leads to leakage (*fuites*) for non-periodic measurements.

4.2.1 Aliasing (*repliment*)

Sampling leads to the impossibility to distinguish high frequency signals from their low frequency alias. A different way to phrase this, is to say that signals at frequencies f , $k f_{sample} - f$ or $k f_{sample} + f$ have the same DFT.

The well-known phenomenon of films showing wheels spinning backwards is related to this problem. The figure 4.2 shows that one cannot distinguish the low frequency sinusoid at f_0 from the high frequency one at $2f_{sample} - f_0$.

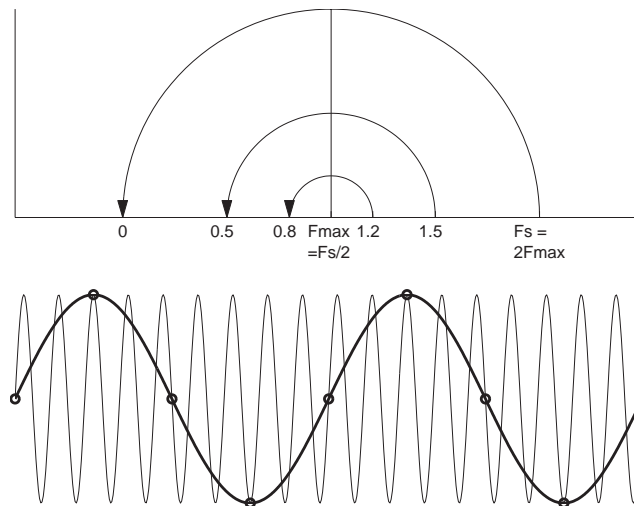


FIGURE 4.2 – Illustration of Shannon's theorem

Shannon's theorem thus states that one must have prior knowledge of the frequency band of the signal to be sure of the actual frequency of the DFT. For example, if the true signal does not have content above

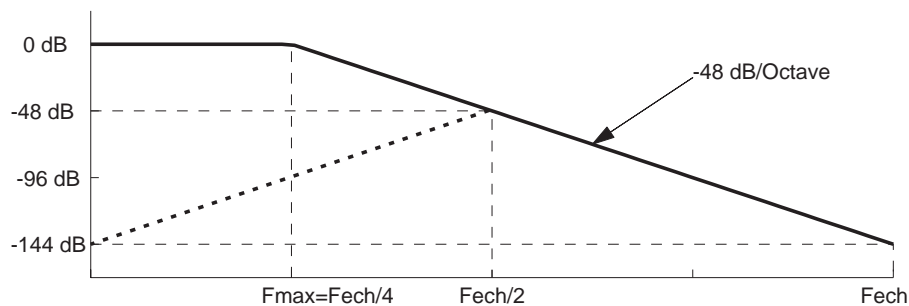
$$f_{max} \leq f_{sample}/2 \quad (4.9)$$

then signal of the DFT below f_{max} cannot be an alias. Note that there are a number of applications (in particular in rotating machinery) where one uses bandpass filters to obtain known aliases of higher frequency signals.

To obtain a correct spectrum estimate, it is thus essential to use a signal with little or no content above $f_s/2$. Theoretically, for a linear system, one could limit the input spectrum. In practice, this can often not be achieved and one prefers the use of low pass filters, called

anti-aliasing filters, that strongly attenuate high frequency components of the signal before digitization thus avoiding aliases.

The procedure to choose the anti-aliasing filter is illustrated in the figure above. Knowing the filter slope (amplitude decrease by frequency octave) and the number of bits of the measurement system, one seeks to obtain an alias of the high frequency spectrum, assumed to be flat, below 1 bit. For a 12 bit system, the level must be lower by $72dB = 20 \log_{10}(2^{12})$. With a slope of 48 dB/Octave, one thus needs to sample at 4 times the last frequency of interest to make sure that the spectrum in the $3f_{\max}$ to $4f_{\max}$ band gets aliased with an amplitude below 1 bit (the signal is assumed to be white).



In practice, modern systems use a fixed frequency anti-aliasing filter. The system thus always samples at its maximum sampling frequency and a specialized DSP (digital signal processor) uses numeric anti-aliasing before sub-sampling to the desired frequency.

It is reminded that filtering is never perfect and that good filters are expensive. There will thus always exist some compromise in aliasing treatment.

4.2.2 Leakage (*fuites*)

Leakage is a difference between the discrete transform obtained in practice and the continuous transform that one seeks to estimate. These errors are associated to the finite length of the measurement. The DFT assumes the signal to be periodic. When this hypothesis is not verified one has leakage errors..

To understand leakage, one can consider discrete sampling over a finite length as the product of the true continuous signal by a rectangular window formed by series of impulses at the time steps during a window length (amplitude 1 for $t_k = k\Delta t$ and 0 elsewhere). In the frequency domain, the transform obtained is thus the convolution of the desired spectrum of the true function by the spectrum of the measurement window, which for a rectangular window is shown below.

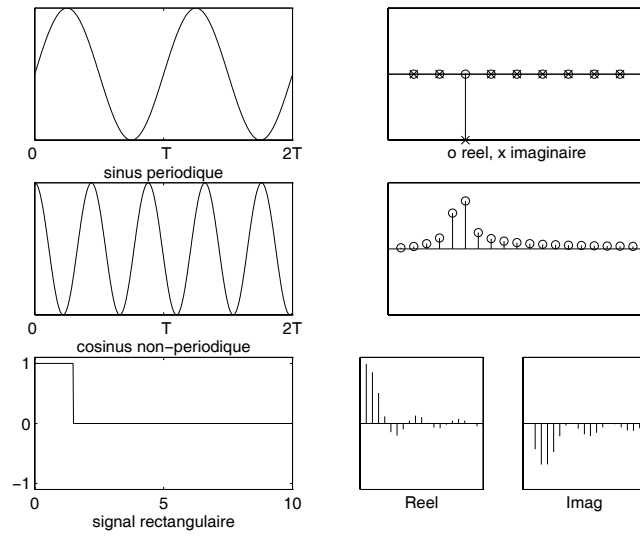
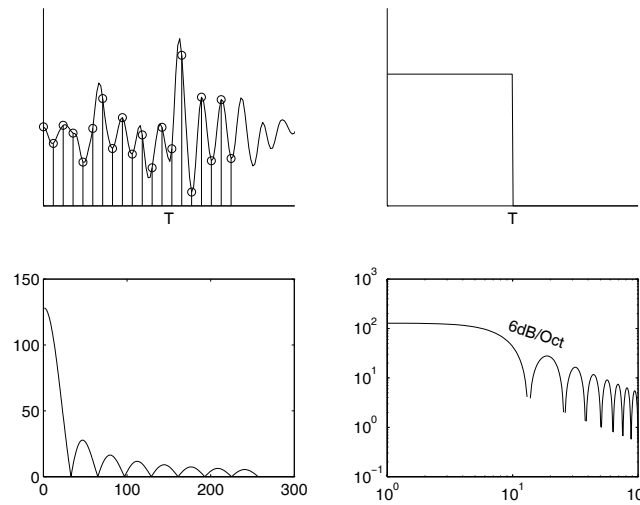


FIGURE 4.3 – Leakage for simple signals

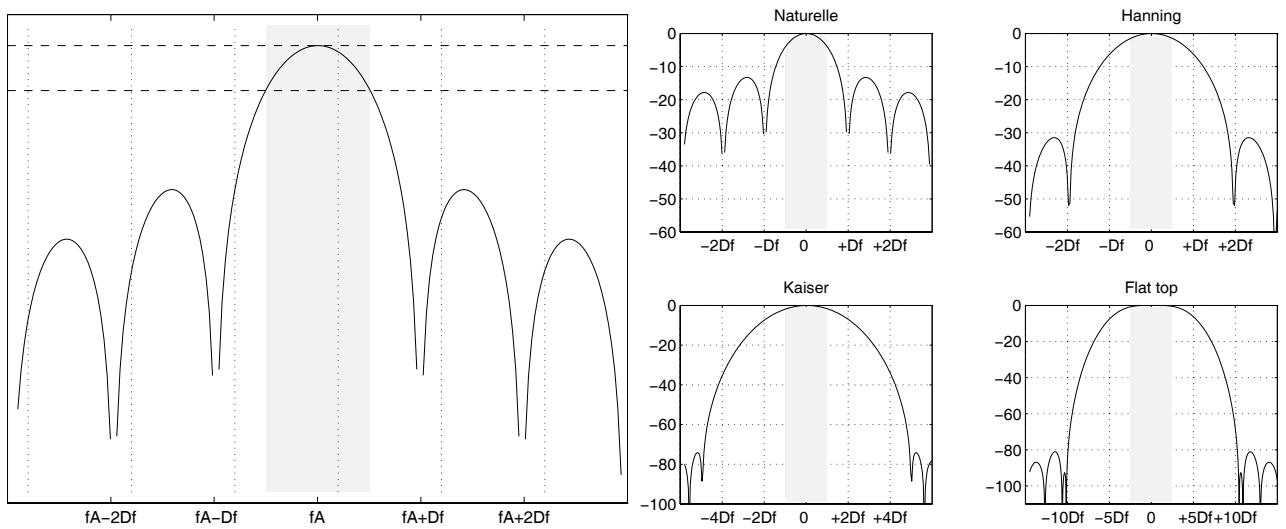


The rectangular window does not, in many aspects, have the ideal spectrum and it is often desirable to use something else. The most common windows are sine combinations

$$w(t) = \left(\sum_{j=0}^3 a_j \cos \left(\frac{j2\pi}{N\Delta t} t \right) \right) \left(\sum_{k=1}^N \delta(t - k\Delta t) \right) \quad (4.10)$$

Type	a_0	a_1	a_2	a_3	Utilisation
Rectangular	1	0	0	0	periodic signals
Hanning	0.5	-0.5	0	0	continuous random signal
Hamming	0.54	-0.46	0	0	continuous random signal
Flat top	0.281	-0.521	0.198	0	Calibration
Kaiser-Bessel	1	-1.298	0.244	0.003	separation of close frequencies
Exponential					Transients of length $> T$

The main criteria for the evaluation of the performance of a window are the maximum level of the second lobe, the maximum attenuation level for frequencies other than discrete frequencies, the coherent gain (loss in the energy of the signal).



4.3 Functions of multiple signals and non-parametric identification

Signals that can be used to identify the system behavior can be categorized as follows

stationary	deterministic pseudo-random random	periodic, quasi-periodic
non-stationary	continuous transient	

For example, to characterize the properties of linear time invariant systems, one can consider shaker tests (with stationary random or pseudo-random signals) or hammer tests (deterministic transient)

4.3.1 Correlation and power spectral density

An experiment is said to be **random** if the results obtained for a series of tests performed in **conditions considered to be identical** lead to results are all different. The result of a single

experiment is thus insufficient to describe the phenomenon. One thus seeks to characterize by statistical means all the responses $x(t)$ possibly associated with the same conditions (same structure, environment, duration). For example, all cobblestone roads are different, they do however have common characteristics (frequency associated with the cobblestone size, ...). The following summarizes basic definitions of statistics on random signals, for more details see Refs. [23, 26, 27, 17].

Other important properties are as follows. A signal is **ergodic** if it is possible to permute averaging over possible realizations and averaging in time. A signal is **stationary** if its statistical properties (mean for example) are time invariant.

For stationary and ergodic signals, key features are the mean

$$\bar{x} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(t) dt \quad (4.11)$$

and the mean quadratic value

$$\overline{\{x\} \{x\}^T} = E(\{x\} \{x\}^T) = \frac{1}{T} \lim_{T \rightarrow \infty} \int_0^T \{x\} \{x\}^T dt \quad (4.12)$$

The variance is defined by $\overline{\{x\} \{x\}^T} - \{\bar{x}\} \{\bar{x}\}^T$ and the RMS (“root-mean-square”) as the square root of the mean quadratic value. For a zero mean signal (often assumed), the variance and mean quadratic values are equal.

Another quantity of major interest for random signals is a measure of the speed at which a variable changes. This is in particular used to estimate the time needed to allow a statistically significant measure. This feature is linked to the **auto-correlation** function, noted $R_{xx}(\tau)$ and defined by

$$R_{xx}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(t)x(t+\tau) dt \quad (4.13)$$

where τ is time delay between two measures of $x(t)$. The *auto* prefix indicates that $x(t)x^T(t+\tau)$ is the product of the same sample at two different instants. The auto-correlation function is only independent of t in the case of stationary random signals.

The Fourier transform of the auto-correlation function is called **power spectral density** (DSP) noted $G_{xx}(\omega)$ and given by

$$G_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau \quad (4.14)$$

One can note the simple relation between the *RMS* response and integrals of the auto-correlation function and the DSP

$$x_{rms} = \sqrt{R_{xx}(\tau=0)} = \left[\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x^2(t) dt \right]^{1/2} = \frac{1}{2\pi} \left[\int_{-\infty}^{+\infty} G_{xx}(\omega) d\omega \right]^{1/2} \quad (4.15)$$

4.3.2 Transfer function estimates

For a linear dynamic system, the relation between inputs $u(f)$ and outputs $y(f)$ is characterized by a transfer function

$$\{y(f)\} = [H(f)] \{u(f)\} \quad (4.16)$$

Pay attention : it is easy to compute transfers between arbitrary signals. But the associated meaning may be difficult. For example transmissibilities (transfer from acceleration to acceleration) depends on the characteristics of excitation and thus modal characterization may be problematic.

The first approach to estimate a transfer is to compute the ratio of the Fourier transform of each signal ($y(f)/u(f)$). The averaging of this estimate

$$[\hat{H}(f)] = \sum_{n=1}^N \left(\frac{y(f)}{u(f)} \right)_n \quad (4.17)$$

presents in particular among other the difficulties (see [28]), the risk that $u(f)$ averages towards 0. In practice, one thus prefers alternative methods based on inter- and auto-spectra. One thus defines the H_1 et H_2 and H_2 estimators by

$$[\hat{H}_1(f)] = \frac{\sum_{n=1}^N y_n(f)^H u_n(f)}{\sum_{n=1}^N u_n(f)^H u_n(f)} = \frac{\sum_{n=1}^N (G_{yu})_n}{\sum_{n=1}^N (G_{uu})_n} = \frac{\hat{G}_{yu}}{\hat{G}_{uu}} \quad (4.18)$$

and

$$[\hat{H}_2(f)] = \frac{\sum_{n=1}^N y_n(f)^H y_n(f)}{\sum_{n=1}^N u_n(f)^H y_n(f)} = \frac{\sum_{n=1}^N (G_{yy})_n}{\sum_{n=1}^N (G_{uy})_n} = \frac{\hat{G}_{yy}}{\hat{G}_{uy}} \quad (4.19)$$

These two estimators provide approximations of H and one defines the correlation coefficient also called **coherence** by

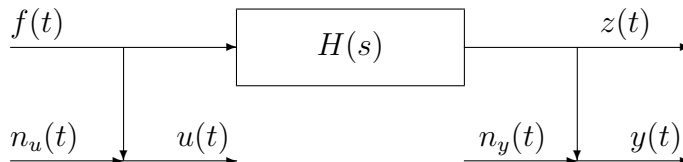
$$\gamma^2 = |\hat{G}_{uy}|^2 / (\hat{G}_{uu} \hat{G}_{yy}) \quad (4.20)$$

By construction coherence varies in the $[0 \ 1]$ interval. A value close to 1, indicates a perfectly linear relation between u and y . A smaller coherence can come from

- measurement noise not correlated to $u(t)$ and/or $y(t)$.
- a non-linearity of the considered system
- leakage problems
- time delays in the system that are not compensated for

On can demonstrate [28] that the variance of the coherence (and that of the estimated transfer) vary with $1/\sqrt{N}$ (number of experiment repetitions), hence the interest of doing averaging.

To give a little more detail, one considers the following diagram where one adds noise to inputs n_u and outputs n_y .



For an uncorrelated noise on input only, the non-correlation leads to $G_{nu} = G_{ny} = 0$, the mean spectra are thus given by

$$\begin{aligned} \hat{G}_{uu} &= \hat{G}_{ff} + \hat{G}_{n_u n_u} \\ \hat{G}_{yy} &= |H|^2 \hat{G}_{ff} = |H|^2 (\hat{G}_{uu} + \hat{G}_{n_u n_u}) \\ \hat{G}_{uy} &= \hat{G}_{xy} = H \hat{G}_{ff} \end{aligned} \quad (4.21)$$

from which comes the fact that the H_2 estimator is unbiased : the estimate is exactly equal to H , the estimator is insensitive to uncorrelated noise on input. For the H_1 estimator,

$$H_1 = \frac{\hat{G}_{yu}}{\hat{G}_{yy}} = \frac{H}{1 + \hat{G}_{n_u n_u} / \hat{G}_{ff}} \quad (4.22)$$

there is a sensitivity that only decreases towards 0 for a large number of averages.

Similarly, one demonstrates that the H_1 estimator is unbiased for output noise only.

4.3.3 Cas multi-variable

Dans le cas de plusieurs entrées/sorties, l'estimateur H_1 est donné par

$$[H_1]_{NS \times NA} = [\hat{G}_{yu}]_{NS \times NA} [\hat{G}_{uu}]_{NA \times NA}^{-1} \quad (4.23)$$

où la matrice $[\hat{G}_{uu}]$ doit être non-singulière ce qui en pratique est obtenu en imposant que les entrées u soient décorrélées.

L'estimateur H_2 serait défini par la résolution de

$$[H_2]_{NS \times NA} [\hat{G}_{uy}]_{NA \times NS} = [\hat{G}_{yy}]_{NS \times NS} \quad (4.24)$$

où l'on voit que pour $NS \neq NA$ un pseudo-inverse devrait être défini, ce qui en pratique limite l'utilisation de l'estimateur H_2 aux cas à une seule entrée ou à ceux où chaque entrée est considérée séparément.

Considérons maintenant un problème avec des bruits sur les entrées et sorties

$$\{y(f)\} + \{n_y\} = [H(f)]_{NS \times NA} (\{u(f)\} + \{n_u\}) \quad (4.25)$$

L'approche de moindres carrés totale (*total least squares*) reformule le problème sous la forme

$$[H \quad -I] \begin{bmatrix} \{u(f)\} + \{n_u\} \\ \{y(f)\} + \{n_y\} \end{bmatrix} = 0 \quad (4.26)$$

La multiplication à droite de cette équation par $\begin{bmatrix} \{u(f)\}^H + \{n_u\}^H & \{y(f)\}^H + \{n_y\}^H \end{bmatrix}$ et le moyennage des termes de la matrice résultante conduit à

$$[H_\nu \quad -I] \begin{bmatrix} \begin{bmatrix} \hat{G}_{uu} \\ \hat{G}_{yu} \end{bmatrix} \\ \begin{bmatrix} \hat{G}_{uy} \\ \hat{G}_{yy} \end{bmatrix} \end{bmatrix} = 0 \quad (4.27)$$

que l'on peut résoudre au sens des moindres carrés. On pourra éventuellement avoir intérêt à résoudre le problème séparément pour chaque sortie (NS problèmes de dimension $NA + 1$ plutôt qu'un problème de dimension $NS + NA$). L'estimateur résultant est appelé H_ν .

Comme toujours, les problèmes de moindres carrés sont sensible aux choix de l'échelle sur chaque variable. Il peut donc être utile de pondérer les entrées ou sorties en fonction du niveau de bruit attendu sur chaque variable.

4.4 Conclusion

Ce chapitre a couvert les notions principales liées à la transformée de Fourier nécessaire à la compréhension pratique du fonctionnement des systèmes d'acquisition utilisés pour l'étude de la dynamique des systèmes.

Pour l'estimation de fonctions de transfert à partir de mesures à contenu spectral large bande H_ν est presque toujours préférable aux autres estimateurs. Si H_ν n'est pas disponible, utilisez H_1 si les sorties sont bruitées où en présence de plusieurs excitations indépendantes. Si l'entrée est bruitée où s'il y a des problèmes de leakage au niveau des résonances, H_2 est préférable.

Le choix des signaux d'entrée n'a pas été évoqué c'est un aspect important des applications pratiques et il convient de se souvenir que les essais en excitation sinus stabilisée donnent de très loin les estimations les moins bruitées des fonctions de transfert d'un système linéaire. Les temps de mesure sont cependant plus long, ce qui a conduit à une certaine désaffection en dehors des applications très basse fréquence ou demandant une très bonne précision (vibrométrie laser par exemple).

L'estimation de spectre fréquentiel n'est pas la seule manière de caractériser un système dynamique. Il est possible de chercher à estimer directement une représentation paramétrique au cours de la mesure (algorithmes d'identification en ligne [28]), d'utiliser directement les paramètres de Markov (voir section 7.5), ou d'utiliser d'autres transformées que celle de Fourier. La transformée en z est une forme de la transformée de Fourier aux problèmes de fonctions échantillonnées (contrôle digital). Les diverses familles d'ondelettes peuvent fournir une autre alternative. Pour plus d'informations, il existe de très bons ouvrages de référence [25, 24, 28].

L'estimation des fonctions de transfert en un certain nombre de points de fréquence par les techniques de traitement du signal est souvent appelée identification non-paramétrique, car en dehors de l'échantillonnage régulier, on ne fait pas d'hypothèse sur la forme de la réponse. Les estimations non-paramétriques peuvent être utilisées directement pour la vérification d'un comportement vibratoire acceptable, la prédiction de la réponse d'une structure légèrement modifiée, ou plus généralement servir de référence pour une identification paramétrique ou le recalage d'un modèle élément fini.

Chapitre 5

Model reduction methods

The term reduced model is used here to signify a model with a sufficient number of DOFs to describe the dynamics of the system

Assuming restrictions on the frequency range of interest and the considered inputs, it will be shown that the true response can be approximated very well within a restricted subspace that can be compute *a priori*. The main classical methods are introduced in section 5.3. They are known as assumed modes methods, Ritz-Galerkin analysis, condensation, component mode synthesis, super-element analysis, domain decomposition, etc. The generic designation *model reduction* methods indicates that the subspace size is sufficient to describe the dynamic behavior properly but whose size is much smaller than that of a FEM model that seeks to represent local geometrical and material detail.

FE modeling can itself be seen as a reduction method as shown in section 5.1.1 but normally the need to account for local geometrical detail or spatial variations of the solution lead to models whose size is significant and often not compatible with analyses performed in structural dynamics (frequency response, time integration over large time intervals, ...)

Section 5.2 motivates classical eigenvalue solvers and shows how they are particular reduction methods. Section 5.4 shows how the classical methods can be derived by objectives on the prediction of the dynamic response to given loads or enforced displacements. And then introduces the concept of error control and methods that can be derived from it.

5.1 Ritz methods

5.1.1 Approximation of a continuous model

Displacement approximation methods (also called *Ritz Galerkin methods* or *assumed mode methods*) seek an approximation of mechanical problems in a subspace of cinematically admissible displacements. Such subspaces are described by a base of time independent displacements $N_n(x)$ and Degrees of Freedom $q(t)$

$$\{X(x, t)\} = \sum_n \{N_n(x)\} q_n(t) \quad (5.1)$$

For static problems, the approximate solution minimizes potential energy within the sub-

space

$$\mathcal{E}_p = \mathcal{E}_i + \mathcal{E}_e = \frac{1}{2} \{q\}^T [K] \{q\} + \{q\}^T \{F_q\} \quad (5.2)$$

or in a more general setting, correspond to a stationary solution for an integral formulation of the principle of virtual works.

The finite element method can be used to treat structures with arbitrary complexity by using geometrically piece-wise solutions on each element. Functions of the approximation subspace are non-zero only very locally around each node. Reduction methods introduced in this chapter seek to build an approximation with base vectors that are defined over the whole model or one of its components. Removing the constraint on the use of local vectors often allows the use of a smaller number of vectors.

Building of Ritz models for simple continuous models will be detailed in class.

5.1.2 Reduction for a discrete model

For models that are already discretized, the equations of motion take the general form

$$\begin{aligned} [Ms^2 + Cs + K]_{N \times N} \{q(s)\} &= [b]_{N \times NA} \{u(s)\}_{NA \times 1} \\ \{y(s)\}_{NS \times 1} &= [c]_{NS \times N} \{q(s)\}_{N \times 1} \end{aligned} \quad (5.3)$$

The principle of a Ritz Galerkin reduction (displacement based) is to seek an approximation of the solution in the subspace spanned by a reduction basis, defined by the real matrix $[T]$, associated with generalized DOFs q_R

$$\{q\}_N = [T]_{N \times NR} \{q_R\}_{NR} \quad (5.4)$$

By itself this assumption leads to an over-determined set of N equations, one thus assumes that the work of these equations is zero. Which translates into the fact that the left product by T^T of the equations of motion with (5.4) assumed is equal to zero. Thus

$$\begin{aligned} [T^T M T s^2 + T^T C T s + T^T K T]_{NR \times NR} \{q_R(s)\} &= [T^T b]_{NR \times NA} \{u(s)\}_{NA \times 1} \\ \{y(s)\}_{NS \times 1} &= [c^T]_{NS \times NR} \{q_R(s)\}_{NR \times 1} \end{aligned} \quad (5.5)$$

For static problems, the verification of the equilibrium conditions thus projected correspond to the minimization of the potential energy on all the fields within the subspace generated by T . In general these fields are cinematically acceptable and the reduction corresponds to the displacement method of continuous mechanics. For the finite element method, shape functions give knowledge of motion at every point in space. For reduction, the basis T of the subspace gives knowledge of motion at every DOF of the unreduced model through (5.4). From there, shape functions allow the determination of any quantity of interest (displacement, deformations, strains, ...).

The accuracy of the reduction is linked to the fact that all states q that are found in reality can be well approximated by a vector within the subspace spanned by T . Thus model reduction methods must somehow build an appropriate subspace using *a priori* knowledge on the responses of interest. The justification of various existing strategies is the objective of this chapter.

In all these notes one assumes that T is real which simplifies the writing of many equations and does not introduce fundamental limitations. The only reason to use complex shape would be further reduction which is not really needed. One also assumes the use of a method that preserves reciprocity : the equations of motion are multiplied by T^T rather than another subspace which is also considered in some FEM formulations.

There is a significant analogy between reduction and displacement based discretization methods.

TABLE 5.1 – Parallel between FE and reduction methods.

	Finite elements	Reduction
Support	continuous domain (triangle, ...)	finite element mesh
Shape functions	polynomial bases	static and dynamic "modes"
Matrix computations	numerical integration	$T^T [K] T$ projection
Assembly	localization matrices /continuity	conditions on interface, CMS
Validity	Fine mesh for solution gradients	appropriate choice of loading

When building FEM models, the main assumption is that the mesh is sufficiently refined (h refinement) or that the element order (p refinement) is sufficiently high on each area to represent variations of the proposed solution. Properties that are ignored or not well accounted for are the properties of other nearby elements (except certain shell formulations) or loading and responses of interest (except for mesh adaptativity methods).

When building reduction methods, the main assumptions are the existence of an precise underlying FEM model and restrictions on the considered loads (spatial and frequency range restrictions in structural dynamics) and parametric variations for a family of models. A standard reason to use model reduction is that the numerical cost of running direct solvers for the FEM model is prohibitive.

For static applications or eigenvalue solvers, one can show that the accuracy of a reduced model increases with the subspace size. In the FE method, the quality of predictions is improved by refining the mesh (h-method) or the order of the approximation within each element (p-method). If the refinement is such that subspace spanned by the unrefined model is included in the refined one, then one has a guarantee of uniform convergence.

When considering a reduction that combines FEM solutions, the ideal situation is to include the solution in the subspace. For a frequency response at one frequency this is for example possible and a simple way to obtain an approximation at nearby frequencies. For time responses or frequency responses over a certain range, the subspace size tends to grow. One thus needs to define a priori solutions that will span an appropriate subspace. This will be done in section 5.3, convergence analysis and error control will be discussed in section 5.3.

5.2 Eigenvalue computation methods

A fundamental distinction must be made between full and partial eigenvalue solvers.

Full eigenvalue solvers seek to compute all eigenvalues of a full matrix. Such solvers are well stabilized and should be accessed through numerical libraries optimized for this purpose. The

most common library for this purpose is LAPACK [29] and microprocessor vendors typically provide tuned versions of this library. Full eigenvalues solvers are typically used for problems involving a few thousand eigenvalues at most. Their performance can be enhanced by taking properties of symmetry, tri-diagonal form, ... of the input matrices.

Readers of these notes are unlikely to need detailed knowledge of full eigenvalue solver methods. More details can be found in chapter 6 of [16] and Ref [30].

Partial eigenvalue solvers are of much more significant interest and will be detailed here. These notes only give a short summary of main results.

5.2.1 Rayleigh quotient and associated properties

The Rayleigh quotient is given by

$$Q(q) = \frac{\{q\}^T [K] \{q\}}{\{q\}^T [M] \{q\}} \quad (5.6)$$

For $q = \phi_j$ the Rayleigh quotient is equal to the eigenvalue $Q(\phi_j) = \omega_j^2$

Eigenvalues correspond to stationary points of Rayleigh quotient. For $\{q\} = \{\phi_j\} + \epsilon \{y\}$ one has

$$Q(q) = \omega_j^2 + \epsilon^2 O(\{y\}) \quad (5.7)$$

The k^{th} eigenvalue corresponds to the minimum of the Rayleigh quotient in the subspace that is orthogonal to the first $k - 1$ eigenvectors.

Rayleigh's theorem on constraints : for a linear system with m additional constraints, the eigenvalues $\tilde{\omega}_j^2$ of the modified system are bounded by those of the initial eigenvalue problem by

$$\omega_j^2 \leq \tilde{\omega}_j^2 \leq \omega_{j+m}^2 \quad (5.8)$$

More info in [16].

5.2.2 Inverse iteration : finding the first mode

Inverse iteration methods consider iterations of the form

$$[K] \{q^{k+1}\} = [M] \{q^k\} \quad (5.9)$$

For an arbitrary starting vector q^0 expressed in modal coordinates as

$$q^0 = \sum_{j=1}^N \alpha_j \{\phi_j\} \quad (5.10)$$

Renormalization, test of convergence.

Convergence : see ratio of first two eigenvalues.

What happens with two equal eigenvalues.

5.2.3 Subspace iteration methods

When considering p vectors, one introduces an orthonormalization step between vectors and sees that the convergence is now driven by that ratio to ω_{p+1}^2 .

5.2.4 Krylov subspaces and Lanczos method

One can show easily that the dynamic stiffness can be expressed as a Mac-Laurin development (one also attributes the development to Neuman)

$$[Ms^2 + K]^{-1} = \sum_{j=0}^p (-s^2)^j (K^{-1}M)^j K^{-1} - (-s^2)^p (K^{-1}M)^{p+1} (Ms^2 + K)^{-1} \quad (5.11)$$

Where it is assumed that the stiffness matrix is non singular (a discussion of strategies used for cases with rigid body modes is done in section 5.3.2). From this exact development, it appears that the finite development of the response to a unit input in the vicinity of $s = 0$ are given by

$$[Ms^2 + K]^{-1} [b] = \sum_{j=0}^p (-s^2)^j [(K^{-1}M)^j K^{-1}] [b] + o(s^p) \quad (5.12)$$

For a case with a single input ($NA = 1$), the vectors are known as *Krylov* vectors. The use of multiple inputs corresponds to the generation of Krylov subspaces. The building of a reduction basis T by combination of multiple Krylov vectors or subspaces is thus natural if one seeks to represent the low frequency response to a set of inputs (forces in this case of mechanical systems).

One classical result is that Krylov vectors are very collinear [31]. A model projected on Krylov vectors would thus have very poor numerical conditioning. The standard procedure is thus to combine the Krylov generation procedure with a vector Orthogonalization scheme. The most classical approach is the conjugate gradient method which in the case of eigenvalue computation methods is known as the Lanczos method [32, 30, 16]. A generalized Lanczos procedure based on blocks is thus given by

$$\begin{aligned} 0) & \quad T_0 = [0]_{N \times NA} \\ 1a) & \quad \hat{T}_{L1} = [K^{-1}]_{N \times N} [b]_{N \times NA} \\ 1c) & \quad T_{L1} = \hat{T}_{L1} Q_1 \text{ with } Q_1^T \left[\hat{T}_{L1}^T M \hat{T}_{L1} \right] Q_1 = [I]_{NA \times NA} \\ \dots & \\ (n+1)a) & \quad \tilde{T}_{Ln+1} = [K^{-1}M] T_{Ln} \\ (n+1)b) & \quad \hat{T}_{Ln+1} = \left[I - T_{Ln} \left(\tilde{T}_{Ln} M \right) - T_{Ln-1} \left(\tilde{T}_{Ln-1} M \right) \right] \tilde{T}_{Ln+1} \\ (n+1)c) & \quad T_{Ln+1} = \hat{T}_{Ln+1} Q_{n+1} \text{ with } Q_{n+1}^T \left[\hat{T}_{Ln+1}^T M \hat{T}_{Ln+1} \right] Q_{n+1} = [I]_{NA \times NA} \end{aligned} \quad (5.13)$$

where the Orthogonalization of step nc can be performed by any appropriate algorithm (Gram Schmidt, Modified GM, iterative GM, ...). It is also important to note that the orthogonalization with respect to the previous two subspaces (step nb) is not always sufficient to guarantee the numerical orthogonality of the retained vectors. This problem has led to a large literature on *numerical* modes in the eigenvalue solvers [16, 30, 33].

5.3 Classical bases for component mode synthesis

This section presents classical bases used in model reduction techniques. The presentation is historical and does not attempt to justify the selection of methods, which will be done in section 5.4.

5.3.1 Modal truncation and static correction

One saw in chapter 3 that the exact dynamic stiffness can be approximated by a series of modal contributions. As in all series, faster computations are achieved by truncating the series. Many FEM software, thus implement *modal methods* where only modes within the considered frequency band are retained (for frequency or time domain simulation).

The pure truncation of high frequency modes is often a poor approximation. One thus introduced the idea that the *quasi-static* contribution of high frequency modes should be retained. Note that this idea also leads to the apparition of a D term in state-space models.

In the modal series (3.45), one can distinguish NM modes within the frequency band of interest for which $s = i\omega$ is close to ω_j and high frequency modes for which $s \ll \omega_j$. One can thus decompose the series in two parts

$$H(\omega) = [c] [-M\omega^2 + K]^{-1} [b] \approx \sum_{j=1}^{NM} \frac{[c] \{\phi_j\} \{\phi_j\}^T [b]}{-\omega^2 + \omega_j^2} + \sum_{j=N_M+1}^N \frac{[c] \{\phi_j\} \{\phi_j\}^T [b]}{\omega_j^2} \quad (5.14)$$

where the second part does not depend on frequency. It is called **static correction** (or residual flexibility) since it assumes that high frequency modes have a quasi-static behavior.

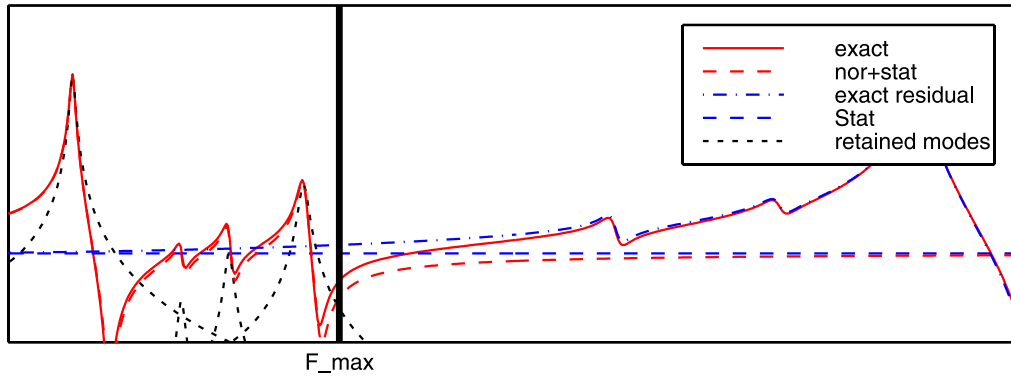


FIGURE 5.1 – Modal and residual contributions

It is not practical to compute all high frequency modes, but for structures without rigid body modes, the orthogonality conditions (3.36)-(3.37) give,

$$T_C = \sum_{j=N_M+1}^N \frac{\{\phi_j\} \{\phi_j\}^T [b]}{\omega_j^2} = [K]^{-1} [b] - \sum_{j=1}^{NM} \frac{\{\phi_j\} \{\phi_j\}^T [b]}{\omega_j^2} \quad (5.15)$$

where it clearly appears that the static correction can be computed by solving a static problem $[K]^{-1} [b]$ (the result is called flexibility) and estimating the NR low frequency normal modes.

For structures with rigid body modes (whose frequency is equal to zero), one must generalize the above expression. One computes the static response of all flexible modes $[K_{Flex}]^{-1} [b]$ using methods detailed in section 5.3.2. These methods of course avoid computation of all modes.

One key result of the approximation of the transfer by modal truncation with static correction is that the response lies within a restricted subspace. One can indeed easily verify that for an observation of all states $c = I$ and T_C given in (5.15), equation (5.14) can be rewritten as

$$q(s) = [\phi_1 \dots \phi_{NM} \quad T_C]_{N \times (NM+NA)} \begin{Bmatrix} \vdots \\ \frac{\phi_j^T b u}{s^2 + \omega_j^2} \\ \vdots \\ u \end{Bmatrix} \quad (5.16)$$

This demonstrates that the proposed approximation can correspond to the use of a Ritz Galerkin basis (see section 3.3.2), combining the target normal models and the residual flexibility

$$[T] = \begin{bmatrix} \phi_1 \dots \phi_{NM} & \left[[K]_{Flex}^{-1} [b] - \sum_{j=1}^{NM} \frac{\{\phi_j\} \{\phi_j\}^T [b]}{\omega_j^2} \right] \end{bmatrix} \quad (5.17)$$

While this basis is often used directly, it is useful to note that the residual flexibility can be replaced by the static response to the considered loads

$$T = [\phi_1 \dots \phi_{NM} \quad [K_{Flex}]^{-1} [b]] \quad (5.18)$$

but since this often leads to fairly collinear vectors, pre-filtering loads so that they do not excite low frequency modes will lead to lower roundoff errors

$$T = \begin{bmatrix} \phi_{1:NM} & [K_{Flex}]^{-1} \left[b - [M [\phi_{1:NM}]] \left[[\phi_{1:NM}]^T b \right] \right] \end{bmatrix} \quad (5.19)$$

In the *component mode synthesis* literature static correction terms are known as *attachment modes*. They have been used in the well-known approaches proposed by Rubin [34] and MacNeal [35].

This modal basis with an augmented state (generalized DOFs associated with the columns of $[K_{Flex}]^{-1} [b]$) is generally preferred to the use of a constant *feed-through* term (D term of the state-space models, corresponding to the second sum in (5.14)) which is an option to represent the residual flexibility.

One can show that the model thus generated has the same low frequency asymptote than (5.14). However, since the mass is also projected this asymptote is associated to a resonance frequency. That frequency has not physical significance, but gives a order 2 rather than 1 development at low frequencies (the low frequency error is proportional to ω^4 rather than ω^2). Some authors have however found useful to neglect the mass (which allows a condensation of the DOFs) and this just comes back to the initial approximation (5.14).

The use of static responses and modes can lead to poor conditioning if the shapes are similar. It is thus usual to orthogonalize the static response with respect to the retain modes. Using the

residual flexibility rather than the flexibility is a possible approach. Another classical solution is to orthogonalize the flexibility terms so that they are both mass and stiffness orthogonal by solving the reduced eigenvalue problem

$$[[T^T K T] - \omega_{jR}^2 [T^T M T]] \{\phi_{jR}\} = \{0\} \quad (5.20)$$

which leads to a new reduction basis $[\hat{T}] = [T] [\phi_{jR}]_{[1:N M + N A]}$. The model with static correction then appears as a truncated series of $N M + N A$ modal contributions with $N A$ (number of inputs) last modes being static correction terms. State-space models used to represent reduced models can thus always be written in the form

$$\begin{aligned} \begin{Bmatrix} \dot{p} \\ \ddot{p} \end{Bmatrix} &= \begin{bmatrix} 0 & I \\ -[\omega_j^2] & -[\Gamma] \end{bmatrix} \begin{Bmatrix} p \\ \dot{p} \end{Bmatrix} + \begin{bmatrix} 0 \\ \phi^T b \end{bmatrix} \{u\} \\ \{y\} &= [c\phi \ 0] \begin{Bmatrix} p \\ \dot{p} \end{Bmatrix} \end{aligned} \quad (5.21)$$

where an assumption on damping is necessary to define Γ [1].

The iterative maximum sequence algorithm [36] is another approach to obtain linearly independent vectors.

The relation between models combining modes and static corrections and older *mode displacement* and *mode acceleration* methods is well described in [37]. The approach is always superior and induces marginal augmentation of numerical costs.

The need for static correction is very case dependent. Single point loads typically require little correction, while relative inputs (typical of the prediction of the effect of internal loads due to modifications, contact, control, ...) are very wrong without it. Given the static correction, one can evaluate its need by computing considering that the series of collocated modal flexibilities is a series of positive numbers that converges to the full flexibility

$$[b] [K]^{-1} [b] = \sum_{j=1}^N \frac{[b] \{\phi_j\} \{\phi_j\}^T [b]}{\omega_j^2} \quad (5.22)$$

which allows an evaluation of the *residual* flexibility as a fraction of the total flexibility.

5.3.2 Static response in presence of rigid body modes

In many problems, the rigid body displacements of the structure are not constrained. The flexible modes, however still have a contribution given by

$$\{q_F\} = [K]_{Flex}^{-1} [b] = \sum_{j=NB+1}^N \frac{\{\phi_j\} \{\phi_j^T b\}}{\omega_j^2} \quad (5.23)$$

To compute this response without computing all normal modes, one seeks to solve the standard static problem $Kq = b$ with K singular. A known result of linear algebra is that this problem has a solution if and only if the equation right hand side b is orthogonal to the solutions of the adjunct linear system. That is

$$[\phi_R]^T [b] = 0 \quad (5.24)$$

where $[\phi_R]$ is a basis for the kernel of K . One call rigid body modes the vectors of this kernel because they correspond to motion that does not induce and strain deformation.

Condition (5.24) expresses that the applied load must be self-balanced (must not excite rigid body modes). Starting from a general load, one can easily build the associated self-balanced load by Orthogonalization with respect to the rigid body modes

$$[b_F] = \left([I] - [M\phi_R] [\phi_R^T M \phi_R]^{-1} [\phi_R]^T \right) [b] \quad (5.25)$$

One then seeks to solve $Kq = b_F$. For that purpose, the method of temporary supports is typically used. One decomposes the structure in two groups of DOFs. DOFs in set I introduce an iso-static constraint, while other DOFs are in set C . An iso-static constraint is by definition such that the rigid body motion is entirely defined by the giving on displacements on the I DOFs

$$[\phi_R] = \begin{bmatrix} I \\ -K_{cc}^{-1} K_{ci} \end{bmatrix} \quad (5.26)$$

For this expression and the orthogonality condition (5.24) one obtains

$$\{b_F\} = \begin{Bmatrix} b_{i,F} \\ b_{c,F} \end{Bmatrix} = \begin{bmatrix} K_{ic} K_{cc}^{-1} \\ I \end{bmatrix} \{b_{c,F}\} \quad (5.27)$$

which can be used to show that the computed response with DOFs I fixed

$$\{\hat{q}_F\} = \begin{Bmatrix} 0 \\ K_{cc}^{-1} b_{c,F} \end{Bmatrix} \quad (5.28)$$

is solution of $Kq = b_F$. Indeed,

$$[K] \{\hat{q}_F\} = [K] \begin{bmatrix} 0 & 0 \\ 0 & K_{cc}^{-1} \end{bmatrix} \{b_{c,F}\} = \begin{bmatrix} K_{ic} K_{cc}^{-1} \\ I \end{bmatrix} \{b_{c,F}\} = \{b_F\} \quad (5.29)$$

The static response thus computed does not exactly correspond to (5.23) because it has components in the direction of rigid body modes. To obtain the desired result, the response is orthogonalized with respect to the rigid body modes

$$\{q_F\} = \left([I] - [\phi_R] [\phi_R^T M \phi_R]^{-1} [\phi_R]^T M \right) \{\hat{q}_F\} \quad (5.30)$$

The procedure to compute the static flexible response is the composed of three steps (5.25),(5.28),(5.30). Figure 5.2 illustrates the static flexible response for an of center load on a free-free plate.

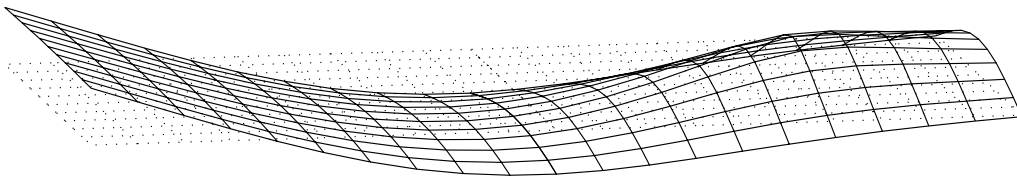


FIGURE 5.2 – Static flexible response of a free-free plate

Computing the static-flexible response exactly is not simple if it is not already implemented in your FEM environment. The easiest alternative is to use a mass shift (compute the response at a non-zero frequency). Thus

$$[K_F^{-1}] [b] \approx [K + \alpha M]^{-1} [b] - \sum_{1, NR} \frac{\phi_{jR} \phi_{jR}^T b}{\alpha} \quad (5.31)$$

only differs from the static flexible response by the modal amplitude of modes such that ω_j^2 is not large compared to α . As long as these modes are retained in the reduction basis, the effect of the response of reduced model is negligible. α can be chosen positive or negative but should not be too small compared to the first frequency to avoid having a response that is too collinear to the rigid body modes. The mass shift technique was originally developed for eigenvalue solvers which are insensitive to such as shift [16]. When model reduction is performed simultaneously to an eigenvalue solve, a mass shifted factor is typically already available.

An alternative to the flexible support method is the use Lagrange multipliers. Since the inverse of the modal matrix is equal to $M [\phi]$ the condition that there is not response along the rigid body modes can be written as $[\phi_R]^T [M] \{q_{Flex}\} = 0$. Thus

$$\begin{bmatrix} K & [M] [\phi_R] \\ [\phi_R]^T [M] & 0 \end{bmatrix} \begin{Bmatrix} q_{Flex} \\ \lambda \end{Bmatrix} = \{0\} \quad (5.32)$$

This problem can be practical in recent software that has a built-in support of Lagrange multipliers in the sparse matrix solver.

5.3.3 Dynamics/static condensation (Guyan)

In the previous section, one introduced static correction to loads. One will now be interested in static responses to enforced displacements.

One divides the model DOFs in two sets actives I and complementary C and builds the Ritz basis associated with unit displacements of the active DOFs resulting from reaction forces applied on the active DOFs only. Such deformations are illustrated in figure 5.3. At a given frequency the exact result is obtained by solving

$$\begin{bmatrix} K_{II} & K_{IC} \\ K_{CI} & K_{CC} \end{bmatrix} \begin{Bmatrix} \langle q_I(s) \rangle \\ q_C(s) \end{Bmatrix} + [Ms^2] \{q\} = \begin{Bmatrix} R_I(s) \\ \langle 0 \rangle \end{Bmatrix} \quad (5.33)$$

where $\langle \rangle$ denotes an enforced quantity.

For the harmonic response at a frequency ω and loads applied on active DOFs I , the exact response can be written using the frequency dependent basis

$$[T(\omega)] = \begin{bmatrix} I \\ -Z_{CC}(\omega)^{-1} Z_{CI}(\omega) \end{bmatrix} \quad (5.34)$$

A projection on this basis has been called *dynamic condensation*. This approach is exact for a given frequency. But using a frequency dependent reduction is of little interest in terms of computational cost. It is however considered in dynamic modal expansion (see section 9.5), non-linear harmonic balance methods [38], periodic structures [39].

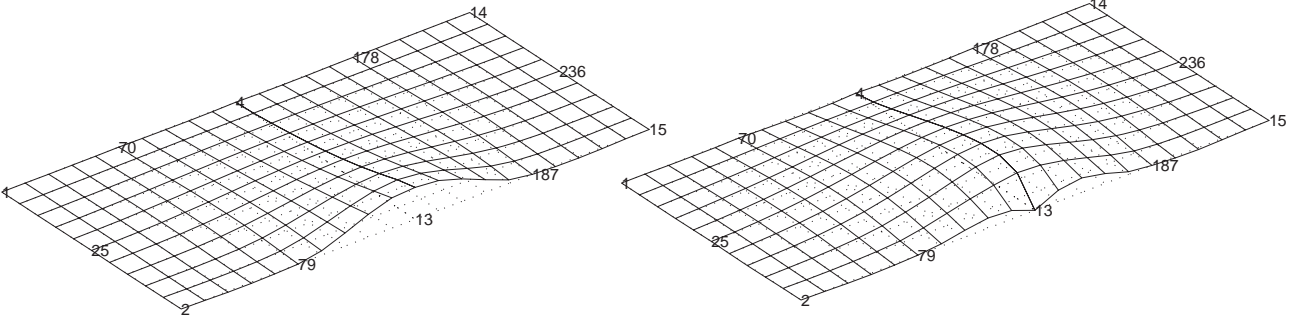


FIGURE 5.3 – Static response to a unit displacement and rotation

Since computing at multiple frequencies is costly, the learning subspace considered for static condensation (also called Guyan condensation [40]) is the exact solution of this problem at $\omega = 0$.

$$[T] = \begin{bmatrix} I \\ -K_{CC}^{-1}K_{CI} \end{bmatrix} \quad (5.35)$$

The columns of T thus computed are called **constraint modes** [41]. In domain decomposition methods, the same responses are called the *Schur complement*.

The projection of the model matrices on the basis thus built is given by

$$\begin{aligned} [T^T K T] &= [K_{II}] - [K_{IC} K_{CC}^{-1} K_{CI}] \\ [T^T M T] &= [M_{II}] - [M_{IC} K_{CC}^{-1} K_{CI}] - [K_{IC} K_{CC}^{-1} M_{CI}] + [K_{IC} K_{CC}^{-1} M_{CC} K_{CC}^{-1} K_{CI}] \\ [T^T b] &= [b_I] - [K_{IC} K_{CC}^{-1} b_C] \\ [c T] &= [c_I] - [c_C K_{CC}^{-1} K_{CI}] \end{aligned} \quad (5.36)$$

The validity of this projection is linked to the assumption that external loads are applied on active DOFs and that inertial loads on complementary DOFs can be neglected.

5.3.4 Static condensation cut-off, Craig Bampton

Since the mass effects are neglected in static condensation the model is wrong when this assumption is not true. A simple problem can be solved to estimate at which frequency this is wrong. One assumes no displacement on the interface and computes *fixed interface modes*

$$\begin{bmatrix} 0 & 0 \\ 0 & Z_{CC}(\omega_j) \end{bmatrix} \begin{Bmatrix} 0 \\ \phi_{j,c} \end{Bmatrix} = \begin{Bmatrix} R_I \\ 0 \end{Bmatrix} \quad (5.37)$$

since at modal frequencies the kinetic and strain energies are equal the assumption that inertia effects can be neglected does not hold.

The first fixed interface mode frequency gives a frequency limit (cut-off frequency) for the validity of static condensation [42]. The figure below shows the first fixed interface modes for a plate with active DOFs on the edges and in the middle. One sees that by adding to center nodes, the frequency is raised significantly, which leads to much better predictions of the free/free frequencies.

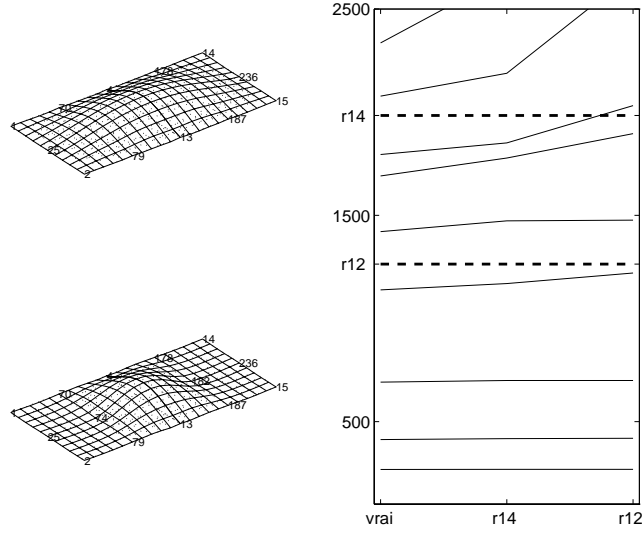


FIGURE 5.4 – Frequency limit illustration

For an efficient static condensation, one should account for the spatial characteristics of applied loads by keeping DOFs associated with

- external loads (actuator locations, etc.)
- interface loads (where to structures are coupled, etc.)
- parametric loads (added masses, location of feed-back forces, etc.)

and possibly account for the desired frequency band by adding additional DOFs to raise the frequency of the first fixed interface mode (to 2 or 3 times the last frequency of interest).

The selection of additional DOF is however not the most efficient method to raise the cut-off frequency. The simplest approach is to keep fixed interface modes as additional Ritz vectors to complement static condensation. One thus considers a basis of the form

$$[T] = \left[\begin{array}{c} I \\ K_{cc}^{-1}K_{ci} \end{array} \right] \left[\begin{array}{c} 0 \\ \phi_{1:NM,c} \end{array} \right] \quad (5.38)$$

which clearly places the reduced model cut-off frequency at after the last retained fixed interface mode (ω_{NM+1}). This approach has mostly been used in component mode synthesis and is known as the Craig-Bampton method[43].

It is useful to note that the stiffness of a Craig-Bampton reduction is block diagonal (the Guyan vectors are orthogonal to the fixed interface modes)

$$\left[\begin{array}{c} 0 \\ \phi_{1:NM,c} \end{array} \right]^T [K] \left[\begin{array}{c} I \\ K_{cc}^{-1}K_{ci} \end{array} \right] = \left[\begin{array}{c} 0 \\ \phi_{1:NM,c} \end{array} \right] \left[\begin{array}{cc} K_{ii} - K_{ic}K_{cc}^{-1}K_{ci} & \\ & 0 \end{array} \right] = 0 \quad (5.39)$$

this property means that the corresponding reduced model has a good sparsity for its stiffness matrix. This property is the basis that makes AMLS (automated multi-level substructuring) algorithms scale well.

5.3.5 Attachment, constraint, interface modes ...

The preceding sections showed that modes needed to be complemented by either attachment modes (static responses to loads) or constraint modes (static responses to enforced displacements). Figure 5.5 illustrates the associated shapes for a simple two plate example.

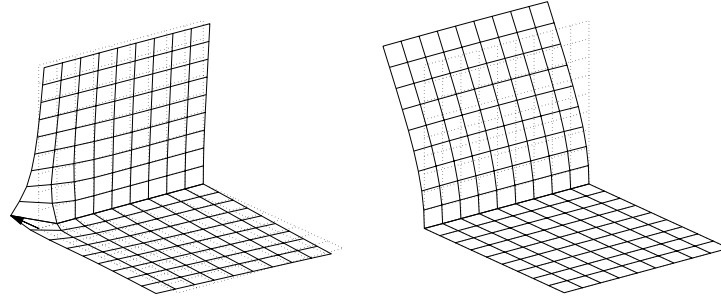


FIGURE 5.5 – Frequency limit illustration

In absence of rigid body modes, the subspace generated by the two approaches are identical since one has

$$\text{vec} \left([K]^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} \right) = \text{vec} \left(\begin{bmatrix} I \\ -K_{cc}^{-1} K_{ci} \end{bmatrix} \right) \quad (5.40)$$

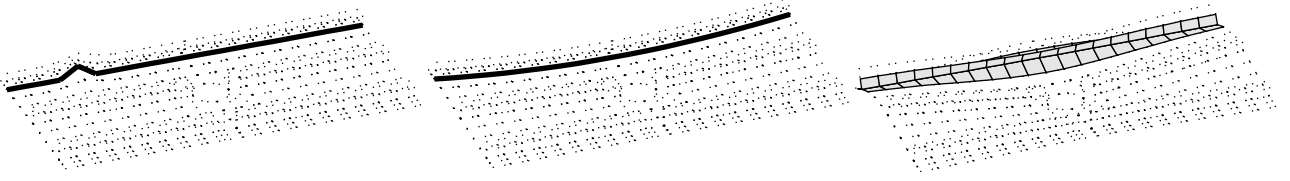
In the presence of rigid body modes, one must complement the subspace of constraint modes by *inertia relief modes*, which correspond to the static deformation induced by a uniform acceleration

$$\text{vec} \left(\begin{bmatrix} [\phi_R] & [K_F^{-1}] \begin{bmatrix} I \\ 0 \end{bmatrix} \end{bmatrix} \right) = \text{vec} \left(\begin{bmatrix} [K_F^{-1}] [M\phi_R] & \begin{bmatrix} I \\ -K_{cc}^{-1} K_{ci} \end{bmatrix} \end{bmatrix} \right) \quad (5.41)$$

One thus sees that the use of methods to compute the static flexible response is not always necessary. The possibility to bypass the problem is even more useful than it might first appear because of many numerical problems detailed in [44].

The number of static modes associated to the DOFs of an interface (computed using constraint or attachment modes) grows rapidly with mesh size and not the dynamic complexity of the structure. This induces a heavy penalty on the numerical cost. Depending on how the components are divided, it can thus be significantly more expensive to solve a problem using substructuring than a direct approach. The literature on mesh partitioning or multi-frontal solvers gives many illustration of this issue.

To remain within classical model reduction methods. The idea of *interface modes* can be introduced. It is a simple redefinition of what a *unit displacement* of the interface might mean. Rather than having unit DOF displacements one can consider unit polynomials on the interface or unit energy motion of a local model containing elements connected to the interface [45].



5.4 Why do these methods work ?

A model is accurate, if it gives a good approximation of the input/output relating for a restricted set of inputs, outputs and parameters

$$[c] [Z(s, \alpha)]^{-1} [b] \{u(\omega)\} \approx [cT] [T^T Z(s, \alpha) T]^{-1} [T^T b] \{u(\omega)\} \quad (5.42)$$

One first starts by motivating standard bases by considering problems with load, in section 5.4.1, and displacement inputs in section 5.4.2.

5.4.1 Representative load inputs

For load inputs, the response is given by

$$\{q(s)\} = [Z(s, \alpha)]^{-1} [b] \{u(\omega)\} \quad (5.43)$$

where the spatial properties of loads are given by the input matrix b and the time/frequency characteristics by the input vector $\{u\}$.

A model reduction method has the objective to represent the relation between $\{u\}$ and $\{q\}$ as accurately as possible. One showed in section 5.2.4 that the basis formed by the sequence of Lanczos subspaces ($T_{Ln}, n \geq 1$) corresponded to a development of the dynamic stiffness around 0. Keeping this basis would thus lead to an “optimal” approximation of the response. It was also shown that this basis contains a good approximation of the low frequency modes and is often used to determine those modes. Inversely, the low frequency modes form a good basis to obtain a good approximation of relation (5.43). If the inputs are described in terms of applied loads, the DOF where these are applied are not constrained, the modes kept are thus often call **free interface modes**.

The basis of free modes does not give an exact static response, one thus normally keeps the first Krylov subspace $K^{-1}b$ whose vectors are called **attachment modes**. This idea is the basis for the *mode acceleration method*, initially proposed by Lord Rayleigh [46], and is used in free interface component mode synthesis methods [35, 34]. One finally notes that the computation of attachment modes for free floating structures was addressed in section 5.3.2.

5.4.2 Representative displacement input

One now assumes that the applied loads can be described by prescribed displacements on a set of DOFs, called interface and noted i (the other DOFs will be called c for complementary). The response to an excitation is given by

$$\{q(s)\} = \begin{Bmatrix} q_i \\ q_c \end{Bmatrix} = \begin{bmatrix} I \\ -Z_{cc}(s)^{-1} Z_{ci}(s) \end{bmatrix} \{q_i(s)\} \quad (5.44)$$

The objective of a model reduction method is to represent the dynamic relation between q_i and q as accurately as possible. In the absence of other information, one assumes that a unit input matrix (q_i of norm 1 at all frequencies) is representative of all loads to be considered and one seeks to use this assumption to build an Ritz basis T .

As for the load input, one considers a development of the dynamic flexibility

$$[-Z_{cc}(s)^{-1}Z_{ci}(s)] = \left[\sum_{j=0}^p [(K^{-1}M)^j K^{-1}]_{cc} + o(s^p) \right] [Z_{ci}(s)] \quad (5.45)$$

The constant term (static) of this development corresponds to a static condensation (evaluation of (5.44) for $s = 0$) which is also called building of constraint modes, static solution to a enforced displacement on the edge, the Schur complement, ...

The use of the Krylov subspace associated with the complementary DOFs does not seem to have been an object of study. This series does however clearly span the subspace of low frequency fixed interface modes ($q_i = 0$) used in the Craig et Bampton [43] component mode synthesis method.

Furthermore, for cases where the interface does not contain the DOFs necessary to enforce an iso-static constraint, one also finds the (*inertia relief modes*) $K_{cc}^{-1}M_{ci}$.

5.4.3 Reduced models and error estimation

For a given reduced model, the standard procedure for error estimation is to build a residual characteristic of error. For model reduction, one assumes that the discretization is correct and thus uses a residual that corresponds to incorrect verification of equilibrium. Thus for harmonic response

$\{q_R\} = [T^T Z(s, p) T]^{-1} [T^T b] \{u(s)\}$ the residual is given by

$$\{\hat{R}(s, u)\} = [Z(s, p)] [T] \{q_r\} - [b] \{u(s)\} \quad (5.46)$$

This residual $\hat{R}_j(s, u)$ is a load characteristic of error. One thus seeks to compute an associated energy. The standard approach is to compute the static response associated with the residual

$$\{R(s, u)\} = [K]^{-1} \{\hat{R}(s, u)\} \quad (5.47)$$

to then define a strain energy error relative to the energy of the response

$$e(s, u) = \frac{\{R(s, u)\}^T [K] \{R(s, u)\}}{\{Tq_R(s, u)\}^T [K] \{Tq_R(s, u)\}} = \frac{\{\hat{R}(s, u)\}^T [K]^{-1} \{\hat{R}(s, u)\}}{\{Tq_R(s, u)\}^T [K] \{Tq_R(s, u)\}} \quad (5.48)$$

with adjustments needed for cases with rigid body modes (see section 5.3.2).

Error estimation is first used to validate the accuracy of a result (the error should be small). But the residual also provides a natural mechanism to iteratively correct the initial reduced model [47]. One thus improves a basis T_i , by considering basis

$$[T_{i+1}] = [[T_i] \quad [R(\omega_k, u_k)]] \quad (5.49)$$

where one must of course limit the number of frequencies and load cases (ω_k, u_k) for which the residual is computed. Without such limitation, the interest of reduction can rapidly disappear.

For the case of error evaluation on modes, it is shown in [47] that this series generates the series of Krylov subspaces. It is however well known that the vectors of this subspace rapidly become collinear (see [31] for example). In practical applications of iteration methods based on residuals, it is thus important to orthogonalize, in the sense of the mass norm, residuals $[R(\omega_k, u_k)]$ with respect to earlier basis $[T_i]$. This is classically done when computing modes with the Lanczos method [16, 33].

5.5 Extending classical CMS methods

5.5.1 Vector sets for reduction

The classical problems used to generate reduction bases are modeshapes and static responses as detailed earlier.

For problems presenting cyclic symmetry [48] or periodicity [49], solutions on a basic repeated cell can be computed at a low cost and then reused to predict more global responses.

It is also possible to compute a limited number of exact responses (in time or frequency) and use the resulting collection of vectors to generate a basis. This is for example used in [50] to predict medium frequency response of a piezoelectric patch.

5.5.2 From vector sets to bases

From a family of vectors, the most common procedure to generate bases is to use the Gram-Schmidt orthogonalisation procedure. [51] thus describes numerically well conditioned implementations.

The MSEQ algorithm [36] proposes an alternative.

Finally SVD based approaches are described in the next section.

5.5.3 SVD, POD and mechanical norms

The singular value decomposition is a classical mathematical tool used to select important directions within a subspace. As detailed in section A.2, the general form for the decomposition of a matrix T is

$$[T]_{N \times NR} = [U]_{N \times N} [\Sigma]_{N \times NR} [V]_{NR \times NR}^T \quad (5.50)$$

where Σ is a diagonal matrix (the elements of the diagonal are called *singular values*, the matrix U contains left singular vectors as columns, the matrix V right singular vectors. U and V are unitary : the singular vectors form bases that are orthonormal with respect to scalar products in the input and output spaces of T .

The classical mathematical definition uses unitary bases with respect to the Euclidian norm. But for applications in mechanics, this norm makes no sense since it does not allow to account for inhomogeneous material properties or different DOF (translation/rotation). It is thus more appropriate to consider variants of the SVD using energy norms [52].

For the classical SVD of a matrix T , right singular vectors V_j are found as solution of the Rayleigh quotient

$$Q_{Euclidian} = \frac{V_j^T T^T T V_j}{V_j^T V_j} \quad (5.51)$$

When considering a collection of vectors T , it thus makes sense to seek stationnary points of

$$Q_{Mechanical} = \frac{V_j^T T^T K T V_j}{V_j^T T^T M T V_j} \quad (5.52)$$

where scaled vectors are assumed to correspond to equal kinetic energy, hence the $T^T M T$ on the denominator and one seeks the lowest strain energy vectors by using $T^T K T$ on the numerator.

In practice from a collection of vectors T , one thus seeks all the modes of the reduced eigenvalue problem

$$[T^T K T - \omega_j^2 T^T M T] \{\phi_{jR}\} = 0 \quad (5.53)$$

and uses the basis $\tilde{T} = [T] [\phi_{jR}]$.

The relation between mechanical norms, the SVD and associated decompositions known as POD (proper orthogonal decomposition, [53] for example) is a critical ingredient of many reduction methods.

Chapitre 6

Reduction and coupled systems

In many applications, reduced component models are available and one seeks to predict the coupled system response. One focuses here on the structure/structure coupling (substructuring, component mode synthesis, domain decomposition, etc.) but other coupling fluid/structure (aeroelasticity, acoustics, ...), electromagnetic fields/structure (active structures, MEMS, ...), thermal/structure are important applications for which reduction and synthesis techniques are very similar.

Reduction and coupling are generally treated as a single problem [54, 41], whereas the point of view of this course is that reduction and coupling can and often should be treated separately.

The point of view developed in this chapter is to consider that coupling is best obtained through a physical interface joining disjoint components. One then introduces perfect links (displacement continuity) as the limiting case, even though this is the historical and still most common application. This change of perspective corresponds to applications of CMS by the authors [55, 56] and allows a clear introduction of locking for incompatible meshes.

6.1 Coupling through a physical interface

As shown in 6.1, one can in a very general maner, consider disjoint dynamic systems (H_1 , H_2 , ..., shown in white) coupled by loads transmitted through an interface H_i . Even though the approach is more general using state-space models, one will focus on second order models characterized by their dynamic stiffness since they are usual in structural dynamics. Since the subsystems are initially uncoupled one can combine the equations of motion by block. Thus for two components, one has

$$\begin{aligned} \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} &= \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \{u(s)\} \\ \{y\} &= [c_1 \quad c_2] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \end{aligned} \tag{6.1}$$

As this represents the decoupled motion, one needs to introduce a coupling relation. Assuming that the interface can be represented by a FEM model, one assumes its DOFs $\{y_j(X, s)\}$ to be linearly related to the component DOFs. Thus,

$$\{y_I(X, s)\} = \sum_{j \in \text{component, interface}} [c_{j\text{int}}(X)] \{q_j(s)\} \tag{6.2}$$

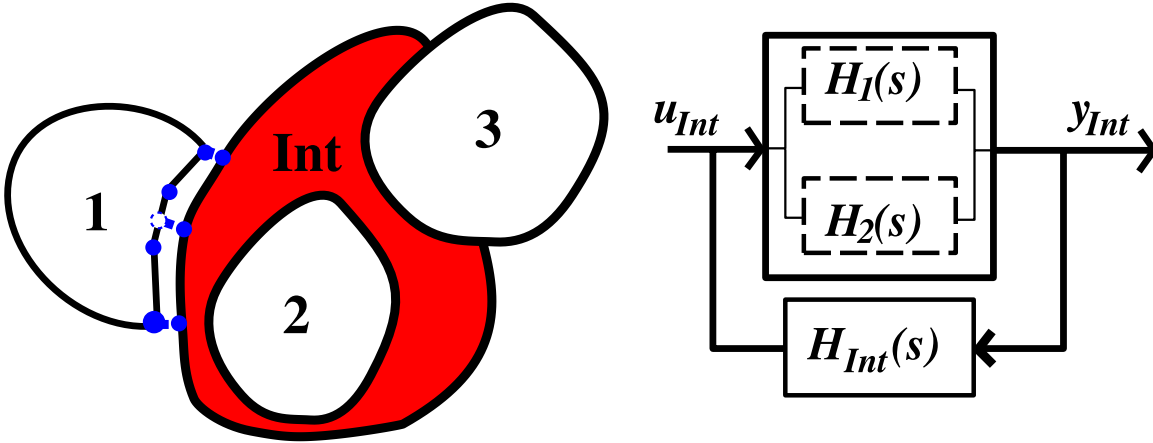


FIGURE 6.1 – Coupling of disjoint model through a finite extent interface

where the coefficients of observation matrices $[c_{j\text{int}}(X)]$ only depend on space. Usually, generalized interface DOFs are simply nodes common between the substructure and the interface. The distinction only becomes important when dealing with incompatible meshes or the reduction of the interface model (see section 6.4).

Having defined the motion of interface DOFs, the equation of motion for the interface is given by

$$\begin{bmatrix} Z_{jj\text{int}} & \cdots & Z_{cj\text{int}} \\ \vdots & \ddots & \vdots \\ Z_{jc\text{int}} & \cdots & Z_{cc\text{int}} \end{bmatrix} \left\{ \begin{array}{l} [c_{\text{int}}] \{q_j\} \\ \vdots \\ \{q_{\text{int}}\} \end{array} \right\} = \begin{Bmatrix} F_{\text{int}} \\ \vdots \\ \{F_{\text{ext,Int}}\} \end{Bmatrix} \quad (6.3)$$

where generalized interface displacements $\{y_{j\text{int}}\} = [c_{\text{int}}] \{q_j\}$ and possible external loads $\{F_{\text{ext,Int}}\}$ are known.

Assuming, to simplify writing, that the interface has no internal DOFs, inserting model (6.3) of the interface loads into the decoupled model (6.1) leads to a model of the coupled system of the form

$$\left(\begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} + \begin{bmatrix} c_1^T & 0 \\ 0 & c_2^T \end{bmatrix} [Z_{\text{int}}] \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \right) \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = [b] \{u(s)\} \quad (6.4)$$

In the trivial case, where all interface DOFs are explicit DOFs of either component, this corresponds exactly to standard FEM assembly by integration of energies over the whole model

$$\left(\begin{bmatrix} Z_{cc}^1 & Z_{ci}^1 & 0 & 0 \\ Z_{i1c}^1 & Z_{i1i1}^1 + Z_{i1i1}^{\text{int}} & Z_{i1i2}^{\text{int}} & 0 \\ 0 & Z_{i2i1}^{\text{int}} & Z_{i2i2}^2 + Z_{i2i2}^{\text{int}} & Z_{i2c}^2 \\ 0 & 0 & Z_{i2c}^2 & Z_{cc}^2 \end{bmatrix} \right) \begin{Bmatrix} q_c^1 \\ q_{i1}^1 \\ q_{i2}^2 \\ q_c^2 \end{Bmatrix} = [b] \{u(s)\} \quad (6.5)$$

The reference to this trivial case should clarify the fact that components are fully characterized by the definition of their DOF set and the interface can be generally defined as the set of elements with DOFs common to more than one set of DOFs.

Automated multi-level substructuring (AMLS, see [57] for example) is a strategy that automates the definition of DOF sets based on graph partitioning considerations and where the notion of interface shown here corresponds to tree branches.

6.2 Coupling through continuity constraints

While the approach above is most general, most of the literature on Component Mode Synthesis has focused on a specific case where the extent of the interface goes to zero. If the extent of the interface is small, the classical case is to define DOFs that are directly comparable (that is the definition of $y_1 - y_2$ is clear). This is the case of so called compatible meshes. Dealing with mesh incompatibility is a difficulty that has been object of many publications (see [58, 45]).

The strain energy associated with a non-zero relative motion, clearly tends to infinity when the extent of the interface goes to zero. One however expects that this energy stays at zero, for zero relative motion. Thus for any proper definition of the problem, one should verify

$$\{y_{1\text{int}} - y_{2\text{int}}\} = [c_1 \quad -c_2] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = 0 \quad (6.6)$$

that is have a discrete approximation of displacement continuity.

In continuous mechanics, a perfect bond is characterized by continuity conditions on displacement (e.g. $x_1(X) = x_2(X)$) which can be written in the general form

$$\{y_1(X)\} - \{y_2(X)\} = \{0\} \quad \forall \quad X \in (\Omega_1 \cap \Omega_2) \quad (6.7)$$

For a discretized model, conditions (6.7) clearly lead to a finite number of constraints which can always be written under form (6.6).

Stress continuity (e.g. $[\sigma_1(X)] \{n\} = [\sigma_2(X)] \{n\}$) is verified for the continuous solution but not taken into account in the weak form of the equations and should thus not be involved in the problem formulation.

Conditions of form (6.6) can be seen as generalized boundary conditions. As the y represent displacements, one talks about cinematic, essential or Dirichlet boundary conditions. In some cases, (see section 6.6), the stiffness can actually be used leading to conditions of the form (6.6)

$$[c_{\text{int}}] [K] \{q\} = \{0\} \quad (6.8)$$

which are then generalized static, natural or Neumann conditions.

It is useful to note that the dual approach to deal with essential conditions of the form (6.6) consists in adding Lagrange multipliers to the decoupled system. One then finds the usual form for mixed formulation FEM models [59]

$$\begin{bmatrix} Z(s) & c_{\text{int}}^T \\ c_{\text{int}} & 0 \end{bmatrix} \begin{Bmatrix} q \\ \lambda \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} \quad (6.9)$$

A classical strategy for the resolution of constrained problems of form (6.9) is to penalize constraints, that is to use $\lambda = c_{\text{int}} q / \epsilon$ with ϵ small. This leads to a coupled problem of the form

$$\left(\begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} + \begin{bmatrix} c_1^T \\ -c_2^T \end{bmatrix} \begin{bmatrix} I \\ \frac{1}{\epsilon} \end{bmatrix} [c_1 \quad -c_2] \right) \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = [b] \{u(s)\} \quad (6.10)$$

where the weighting clearly appears as an interface stiffness going to infinity as ϵ tends to 0. The selection of a proper weighting is often difficult, in particular for non-homogeneous components so that the selection of a layer of elements proposed in the previous section seems more appropriate.

6.3 Numerical handling of constraints

Continuity constraints of CMS problems (Dirichlet type boundary conditions in their full (6.6) or reduced (6.15) formulations), and many other applications, lead to problems with linear constraints. One thus seeks to solve problems of the general form

$$\begin{aligned} [Ms^2 + K] \{q(s)\} &= [b] \{u(s)\} \\ \{y(s)\} &= [c] \{q(s)\} \\ [c_{int}] \{q(s)\} &= 0 \end{aligned} \quad (6.11)$$

Linear constraints can be treated by elimination (one talks about direct or primal methods) or through Lagrange multipliers (dual method). The relative effectiveness of the approach is very much case dependent. The FETI method [60, 61] is for example a dual method that was shown to be effective for use on parallel machines.

Even though the full form of continuity constraints (6.6) is largely acknowledged, most of the literature on CMS focuses on reducing models so that constraints can be eliminated explicitly. One thus uses fixed interface and constraint modes [43], free normal and attachment modes [35, 34] or introduces specific strategies for mixed cases [62]. The idea strongly recommended by the authors is to prefer a numerically robust strategy for constraint elimination that will deal with all combinations of reduced models.

A constraint elimination procedure builds a basis for the subspace generated by the kernel of constraints

$$\text{vect}(T) = \ker(c_{int}) \quad (6.12)$$

The solution of the constrained problem is then found using a Ritz-Galerkin approach where one seeks the response within the subspace $\{q\} = [T] \{q_R\}$ and assumes that the equilibrium equations are verified for a projection on the dual subspace T^T . Thus

$$\begin{aligned} [T^T M T s^2 + T^T C T s + T^T K T] \{q_R\} &= [T^T b] \{u\} \\ \{y(s)\} &= [c^T] \{q(s)\} \end{aligned} \quad (6.13)$$

Such constrain elimination is available in most FEM software to deal with multiple point constraints (MPCs). The following factors were found to be important in the development of the `fe_coor` algorithm implementing this strategy in [63]

- Use of blocs to optimize sparsity : when the constraint only applies to two DOFs, the associated vector has to unity values on the the projection corresponds to addition of the corresponding terms. This is actually the standard FEM assembly strategy. More generally, when constraints only couple some DOFs, precision and speed is gained by obtaining null space vectors having non-zero values only on the blocks linked by constraints.
- Deal with constraint independence : it is often difficult and not desired to guarantee that after reduction constraints will remain independent. For conditions defined on non-reduced models, this allow to avoid treating all interfaces as junctions between two components (there are redundancies for points common to more than two components). For reduced models, the projection (6.6) on reduction bases leads to (6.15) who are only independent if all possible deformations are present. A QR or SVD decomposition of the matrix allow a good determination of the effective number of independent constraints and the building of an appropriate kernel basis. In some cases combining shells or solids, the number of DOF in a block can

however become very large and lead to costly decompositions. The use of pivot sizes in an LU decomposition can then be used as an indication of constraint independence. But this can lead to a deterioration of numerical precision.

The literature on null space determination is significant. Ref [64] for example is associated with open source MATLAB code.

In the CMS literature [41], most authors define an ad-hoc elimination associated with their reduction procedure so that the projected model (6.13) can be written explicitly. The (very) small computational time advantage is not worth the flexibility allowed by fully decoupling the reduction and coupling phases with the general elimination proposed.

It is however necessary to remember that arbitrary reduction can lead to locking phenomena illustrated in section 6.5. Finally, the only difference between Lagrange and elimination procedures is associated with numerical conditioning.

6.4 Discrétisation des interfaces

Que l'on considère une liaison parfaite ou une interface d'extension finie, la validité de la prédiction couplée dépend en grande partie du choix des déplacements d'interface.

Une première approche, que l'on dit ponctuelle, consiste à choisir des déplacements de points particuliers. Le cas classique correspond aux maillages compatibles. On prend alors les DDL des noeuds communs à deux sous-structures (pour une liaison parfaite) ou communs aux sous-structures et à l'interface (dans le cas général). L'extension la plus simple consiste à utiliser les fonctions de forme des éléments liés au modèle de composant pour construire des noeuds intermédiaires. La figure 6.2 montre ainsi un exemple de jonction de maillages solide/plaque avec des éléments de tailles différentes.

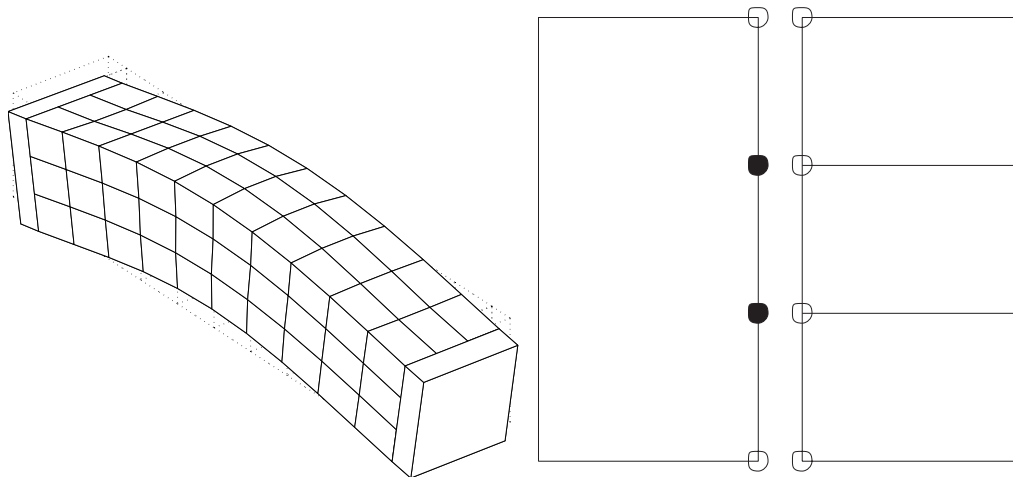


FIGURE 6.2 – Utilisation de déplacement de noeuds de maillage et/ou de noeuds intermédiaires comme déplacement généralisés d'interface. Exemple d'application à un maillage incompatible [65] mais conforme.

Différents auteurs (dont Ref. [66]) considèrent les conditions de continuité (6.7) sous forme

intégrale. La solution discrétisée est alors obtenue par orthogonalisation par rapport à un espace vectoriel de dimension finie (dont les éléments représentent des forces appliquées à l'interface et G est l'ensemble des points communs)

$$\int_{\partial\Omega} \lambda(X) (y_1(X) - y_2(X)) dX = 0 \quad \forall \lambda \in V_\lambda \quad (6.14)$$

Pour $(y_1(X) - y_2(X))$ et λ pris dans des sous-espaces de dimension finie, cette approche conduit clairement à un nombre fini de conditions de la forme (6.6). Cependant, les déplacements d'interface associés sont des combinaisons linéaires de tous les DDL conduisant à un déplacement des points de $\partial\Omega$. Les mêmes DDL interviennent dans la définition, utilisée par l'approche ponctuelle, des déplacements nodaux des points de maillage et des points intermédiaires. En pratique toute formulation intégrale peut donc être considérée comme une forme pondérée d'une formulation ponctuelle. La principale utilité d'une telle pondération est de permettre de relaxer des contraintes et donc dans certains cas d'éviter les phénomènes de verrouillage mis en évidence en section 6.5.

L'idée d'utiliser un espace vectoriel de forces d'interface conduit naturellement à considérer un espace dual de déformations distribuées de l'interface. Il s'agit alors de combiner plusieurs déformations de l'approche ponctuelle pour créer une déformation généralisée ayant des valeurs non nulles sur la plupart de noeuds de l'interface. Si la construction d'un tel sous-espace permet de représenter l'ensemble des déformations effectivement trouvées, on peut réduire les modèles de composant en conséquence (modes de contrainte généralisés [45]). Cette réduction des représentations d'interface est particulièrement utile pour des applications où une grande fraction des DDL de chaque composant sont liés à l'interface (jonctions de solides ou de plaques). Les bases de fonctions de forme élément fini seraient un choix classique pour l'espace vectoriel de déformations d'interface. On peut ainsi (voir figure 6.3) considérer des fonctions de forme de poutre pour représenter les déformées d'une interface entre plaques. Une idée, plus efficace en terme de construction d'un modèle réduit de dimension minimale [67, 45], est d'utiliser les modes d'un modèle local de l'interface (modèle contenant les éléments directement reliés à l'interface comme dans l'exemple de la figure 6.3). Comme indiqué en Ref. [45], le calcul des modes de ce modèle local correspond à une sélection de directions principales ce qui explique l'efficacité constatée de la méthode.

6.5 Réduction, conformité et verrouillage

Dans le cadre des méthodes éléments finis, on sait depuis longtemps que les conditions de saut (6.2) étant définies sur des ensembles continus et donc infinis de points, il n'est pas forcément facile ni souhaitable d'assurer l'équivalence complète entre la condition discrète (6.6) et la condition continue (6.2). On a ainsi démontré l'utilité voire la supériorité d'éléments, dits non conformes, ne vérifiant pas toutes les conditions de continuité attendues. Connaissant le parallèle entre réduction et discrétisation, on se doit donc de prendre en compte, dans le cadre des méthodes de réduction de modèle, la conformité et ses effets sur la convergence (voir Refs. [67, 68, 45]). Dans le cadre des méthodes de réduction, on peut avoir non conformité des éléments, des maillages, mais aussi des représentations réduites des déformations possible de l'interface. On cherche par exemple à prédire les fréquences des trois premiers modes du

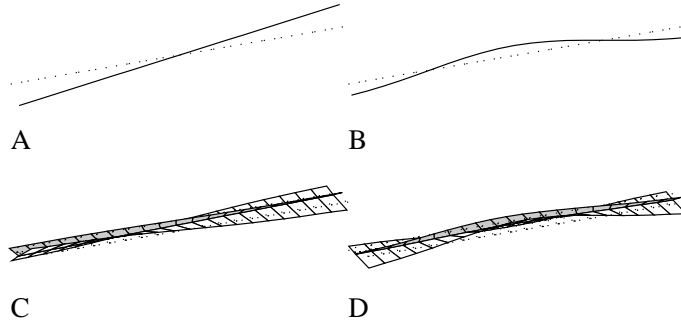


FIGURE 6.3 – A-B déformations généralisées liées aux fonctions d’interpolation d’éléments de poutre. C-D déformations généralisées liées à un modèle local.

panneau raidi montré en figure 6.4.

La base retenue pour le modèle réduit couplé contient, pour chaque composant, les 6 modes de corps rigide et les 2 modes de torsion et flexion montrés sur la figure. L’évolution des fréquences, partie D de la figure, montre clairement le phénomène de verrouillage quand on se rapproche du cas limite de la liaison parfaite (les fréquences du modèle réduit tendent vers l’infini quand la largeur de l’interface tend vers 0). Le mode de torsion converge assez vite vers la valeur exacte (modèle complet). Ce n’est pas le cas des deux modes suivants (flexions), la flexion retenue pour le composant non raidi possède des propriétés de symétrie qui n’existent pas pour le composant raidi (voir partie C de la figure). Une bonne convergence demanderait de retenir plus de modes ou d’utiliser des conditions aux limites chargées en raideur pour le composant non raidi.

Le verrouillage lié à une interface parfaite s’explique de la manière suivante. Les conditions aux limites généralisées (6.6) sont déterminées avant réduction des modèles de composant. On a ici 4 noeuds associés à 6 conditions de continuité. La projection des conditions aux limites sur les bases de réduction des composants conduit à

$$[c_{1\text{int}}T_1 \quad c_{2\text{int}}T_2] \begin{Bmatrix} q_{R1} \\ q_{R2} \end{Bmatrix} = 0 \quad (6.15)$$

Or les déformées d’interface retenues (restriction des modes de corps rigide, de torsion et de flexion à l’interface, 8 déformées par composant) ne sont pas équivalentes pour les 2 composants. Il en résulte que la projection (6.15) contient 10 conditions indépendantes. Sur un modèle découplé de 16 DDL (8 pour chaque composant), l’élimination exacte des conditions (6.15) conduit donc à un modèle à 6 DDL correspondant aux modes de corps rigide. L’incompatibilité des déformées d’interface empêche d’obtenir un mouvement flexible si la continuité exacte est vérifiée. Comme dans le cadre élément fini, ce problème de verrouillage peut être traité en construisant une réduction conforme des modèles ou en acceptant une continuité au sens faible (relaxation de certaines contraintes).

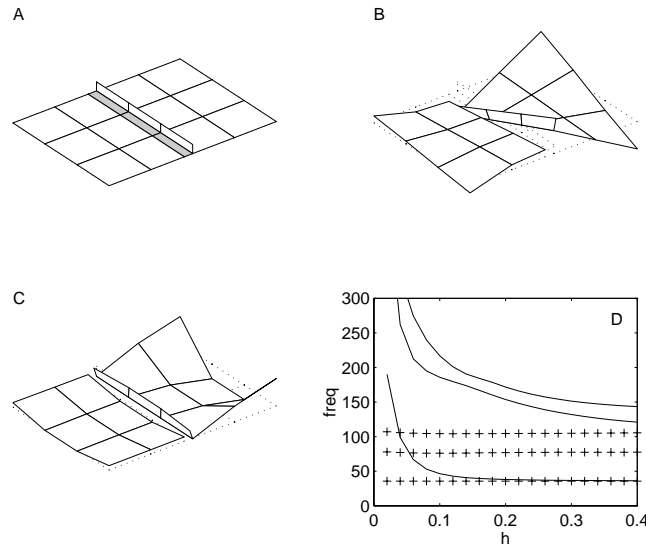


FIGURE 6.4 – Verrouillage d’interface pour un panneau raidi. A zone d’interface en grisé. B-C modes retenus de torsion et flexion des composants. D évolution des 3 premières fréquence du modèle (+) complet et (—) réduit en fonction de la largeur h de la zone d’interface.

Pour construire des modèles conformes, on définit ici [67, 45] des modes de contraintes généralisés comme étant les déformations statiques du composant à des forces appliquées uniquement à l’interface et associées à des déformations généralisées d’interface unitaires. Pour une base de déformées généralisées d’interface commune à deux composants, les bases de modes de contrainte généralisés conduisent à des modèles conformes et ne présentant donc pas de problèmes de verrouillage. Cette approche correspond au relèvement statique de déformées données à l’interface seulement. La méthode de modes de branche [69] est similaire mais se restreint à considérer le composant principal comme une interface (extension statique dans les branches seulement). Le point crucial des applications proposées [67, 45] est donc l’introduction de méthodes de sélection de déformées d’interface.

La relaxation de contraintes est plus complexe car il faut définir ce qu’est une discontinuité acceptable. L’utilisation, considérée ici, d’interfaces ayant une extension et des propriétés physiques est une voie d’étude possible. Ainsi dans l’exemple de la figure 6.4, on observe une bonne convergence du premier mode dès que la largeur de l’interface dépasse 0.2 m soit les 2/5èmes de la largeur maximale de la première rangée d’éléments. Par contre dans l’exemple du panneau raidi étudié en Ref. [45], l’étendue de l’interface (ressorts représentant les rivets de connexion entre panneau et raidisseurs) est trop faible pour réduire les effets de verrouillage de manière significative (la pénalisation est trop forte).

Même si les conditions initiales (6.6) l’étaient, les conditions d’interface réduites (6.15) ne sont souvent plus toutes indépendantes. Le traitement de conditions redondantes est tout à fait possible en prenant quelques précautions algorithmiques particulières (voir section 6.3), il ne donne cependant de bons résultats que si l’on évite les phénomènes de verrouillage illustrés ci dessus.

6.6 Condition aux limites cinématiques et condensation

En mécanique des milieux continus, les conditions de nullité ou de continuité des contraintes normales ($\sigma n = 0$) sont appelées conditions aux limites statiques, naturelles, ou de Neumann. Dans le cadre d'une méthode d'approximation en déplacement leur vérification n'est pas demandée. Pour un espace vectoriel de champs cinématiquement admissibles, leur prise en considération conduit à une minimisation sur un espace plus restreint et donc à une détérioration de la qualité des prédictions.

Les fonctions de forme utilisées pour la formulation élément fini et les lois de comportement permettent d'évaluer les contraintes en tout point et donc de construire des conditions de continuité de contrainte de la forme (6.6). La continuité des contraintes n'étant souvent pas assurée par les éléments, l'imposition directe d'une continuité exacte peut conduire à un phénomène de verrouillage (pas toujours cependant [70]).

Comme pour la continuité des déplacements, on peut éviter le verrouillage en imposant la continuité de manière faible. L'idée généralement retenue (méthode de MacNeal [35] par exemple) est d'imposer la continuité des contraintes généralisées associées à la discrétisation et données par le produit $[K] \{q\}$. A une condition cinématique généralisée $[c_{\text{int}}] \{q\} = 0$, on associe donc la condition statique duale correspondant à l'équilibre des contraintes généralisées et donnée par

$$[c_{\text{int}}] [K] \{q\} = 0 \quad (6.16)$$

Il est toujours possible d'introduire une transformation bijective des coordonnées q telle que la transformée de la matrice $[c_{\text{int}}]$ soit l'identité pour les I premiers DDL et nulle pour les C autres (voir section 6.3). La condition (6.16) est alors donnée par $K_{ii}q_i + K_{ic}q_c = 0$ d'où l'on tire

$$\begin{Bmatrix} q_i \\ q_c \end{Bmatrix} = \begin{bmatrix} -K_{ii}^{-1}K_{ic} \\ I \end{bmatrix} \{q_c\} \quad (6.17)$$

où un lecteur averti reconnaîtra une condensation statique ou de Guyan [40] des DDL d'interface q_i .

L'utilisation souvent faite de conditions sur l'équilibre des forces à l'interface [35, 34, 71, 33] correspond donc à une condensation statique de certains (ou de tous les) DDL d'interface. Comme pour la condensation statique cette approximation n'est valable que si les forces d'inerties associées à ces DDL sont effectivement négligeables.

Au niveau du modèle continu, la condition de continuité des efforts normaux est basée sur le principe d'équilibre de l'action et de la réaction. Si, au lieu de discrétiser cette condition continue, on appliquait le principe au niveau des modèles discrets, on devrait clairement introduire une condition sur l'équilibre dynamique des efforts généralisés. Ceci conduirait à des conditions de la forme

$$[c_{\text{int}}] [Z(s)] \{q\} = 0 \quad (6.18)$$

dont la prise en compte correspondrait à une condensation dynamique. L'élimination des DDL dans une condensation dynamique exacte introduit un modèle ayant une dépendance beaucoup plus complexe à la fréquence et dont la réponse correspond exactement à celle du modèle avant

réduction [72]. Des conditions de la forme (6.18) ne présentent donc pas d'intérêt pratique sauf si elles sont pris en compte de manière approximative. La condition (6.16) correspond ainsi à une approximation quasi-statique de l'équilibre dynamique (6.18).

Chapitre 7

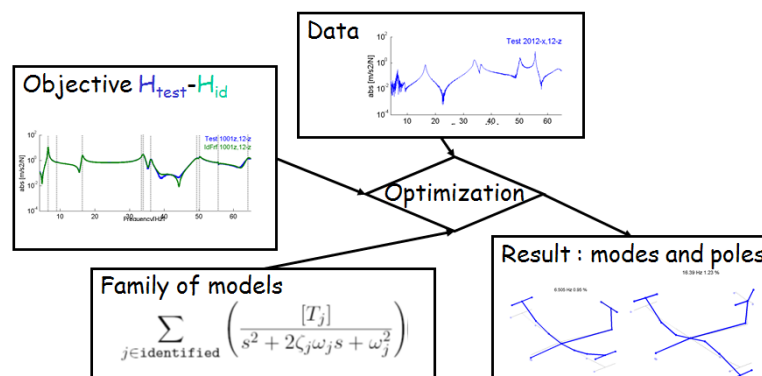
Parametric identification (experimental modal analysis)

After setting up the measurement system, signal processing, and non parametric identification in chapter 4, the building of a low dimension parametric model is the third important step of **experimental modal analysis**. This step seeks to estimate poles (frequencies and damping ratio), modal observabilities and commandabilities. These quantities are then used directly for new predictions or indirectly for model validation, updating (chapter 12), or the construction of hybrid models (analytical modes with test derived damping ratio, structural dynamic modification SDM, ...)

This chapter seeks to give a perspective on the main classes of identification methods. Section 7.1 gives a classification. Section 7.2 summarizes one mode methods, which were developed first and how modern variations remain quite useful. Section 7.3 details the solution of the least squares problem associated with the frequency domain pole/residue model. This model is most often used to account for the residual effects of out of band modes. Section summarizes fundamentals of methods based on polynomial parametrizations 7.4, and section 7.5 those of subspace methods.

In practice, more that one algorithm are often coupled as will be detailed in chapter 8. For further reading, I suggest the following texts [25, 73, 74] or from SDTools [75].

7.1 Classification



This section seeks to give criteria to classify existing identification methods based on

- the data used as reference (section 7.1.1)
- how the model is parameterized (pole/residue, modal model, state-space, rational fraction (ARMA, ...)) as detailed in section 7.1.2,
- the number of considered poles (SDOF single, MDOF multiple). Historically, **SDOF** methods (single degree of freedom, by opposition to MDOF) used a local frequency domain approximation by approximating the frequency response by the contribution of a single pole. They are no longer really useful except to get initial estimates (section 7.2) or for use in mode appropriation methods (*sine-dwell*, *force appropriation*, see section 8.1.2).
- the number of considered inputs and outputs (SISO, SIMO, MIMO). Systems with more than one simultaneous input and output are called MIMO (by opposition to SISO, SIMO or MISO). Accounting for transfer function matrices rather than vectors is often difficult. MIMO test however have significant advantages in their ability to separate close modes. The ability of an identification method to create global models in MIMO configurations is thus an important selection criterion (see section 8.3). It is also common, to use measurement sequences (distinguished by changes in the sensor/shaker configuration). These slight modification of the structure correspond to multiple SIMO test and considering them as MIMO often degrades the accuracy of global methods. The proper procedure is then to use a method for each group of simultaneous measurement, before adjusting the groups. This has been done, I just don't have a good reference on hand.
- the use of frequency or time domain objective functions and the definition of associated optimization algorithms.
- the local or global nature of estimated parameters. For the model of a vibrating structure, poles λ_j are common to all transfers, modal observabilities $c\psi_j$ only depend on sensor positions and nature, modal controllability $\psi_j^T b$ only depend on shakers. Methods that account for the dependencies are called **global**. If the estimation is done for each transfer separately, one obtains results that are slightly incoherent. Patching up the results can be quite difficult, but if possible, it gives an indication of coherence (or variance). The rank constraint on residues (see section 8.3.2) is an illustration of such an approach.

7.1.1 Reference data

The selection of data characterizing the system imposes constraints on the identification process. Time data contain the most *accurate* information but can be quite noisy and correspond to very large volumes of data. One thus mostly use them for structures or excitations that are either non stationary or non-linear.

Signal processing tools leading to the estimation of transfer functions (see chapter 4) allow a major reduction of the amount of data actually used for the identification and allow for a characterization of the system with a low noise level. The computer intensive parametric identification is thus performed on smaller data sets with less noise. Another positive side of transfer functions is that, for structural vibration, the precision of a model is best judged in the frequency domain. The non-parametric identification step is thus generally needed.

Many algorithms are defined in the time domain, but use the inverse Fourier transform of frequency domain results. Subspace methods (see *seridera*) use impulse response functions or

Hankel matrices instead of transfer functions.

Finally, covariance of mixed signals (auto and cross-correlation function discussed in section 4.3.1) are used output only methods.

7.1.2 Parametrizations

A critical aspect of identification algorithms is the form of the considered parametric model. For applications in structural dynamics, the reference model is the **rational fraction** decomposition using first order fractions (also called **pole/residue** model)

$$[\alpha(s)] = \sum_{j \in \text{identified}} \left(\frac{[R_j]}{s - \lambda_j} + \frac{[\bar{R}_j]}{s - \bar{\lambda}_j} \right) + [E(s)] \quad (7.1)$$

where the residue matrix $[R_j] = \{c\psi_j\} \{\psi_j^T b\}$ is given by the product of a column observability $\{c\psi_j\}$ and row controllability $\{\psi_j^T b\}$. When using this form of the residue, one talks about a **modal model**. In the literature, the residue is often written $[A_j] = \{\psi_j\} \{L_j\}$ where modeshape and observability are thought to be the same and the controllability is called modal participation factor L_j . The notation used here, while less common, emphasizes the duality between inputs and outputs.

Residual terms $E(s)$ are characteristic of the transfer being considered (high frequency modes contribute a constant for displacement measurements and a Es^2 contribution for acceleration, ...). The upper residual corresponds to residual flexibility introduced in (5.14). The lower residual corresponds to inertial effects of low frequency modes and can be used to measure mass properties.

All the parameters of this model are **objective** : they are uniquely defined by the choice of the system and the test configuration. For other parametric models, the choice of parameters is not necessarily unique and this affects the accuracy of approximate results. For state-space models, a bijective state transformation $\{\hat{x}\} = [T] \{x\}$ keeps the dynamic input/output relation invariant.

Most software packages use the pole residue model to estimate residuals (see section 8.3) and for some for the resolution of the non-linear least-squares problem (see section 8.3.1).

State space models (3.1) lead to many identification algorithms. One will in particular distinguish sub-space methods that use principal components of Hankel matrices (see section 7.5).

Rational fractions are often considered in control theory. A general form for such models is [28]

$$A(q)y(t) = \frac{B(q)}{F(q)}u(t) + \frac{C(q)}{G(q)}e(t) \quad (7.2)$$

where $A(q)y(t)$ corresponds to the discrete time model $a_0y(t) + a_1y(t - \delta t) + \dots + a_{n_a}y(t - n_a\delta t)$. The most common time domain algorithms are ARX (A and B polynomials), ARMA (A and C), and OE (output error, B and F).

For problems with multiple inputs and outputs, one can consider polynomials with vector or matrix coefficients. One can thus categorize algorithms depending on the fact that A is a scalar or a matrix (see section 7.4 where one describes the LSCE, Ibrahim, ARMA, ... methods).

Finally second order models (3.2) are sometimes used with success although the justification of their use is typically doubtful (see section 7.4.3).

7.1.3 Objectives, algorithms

Identification is always performed through the minimization of a distance between the parametric model and the measured data (or resulting from a non-parametric identification phase). For the handling of a large number of poles and transfers, one almost exclusively uses least-squares problems for the general form

$$\{x\} = \arg \min_{\{x\}} \|[A] \{x\} - \{b\}\|^2 \quad (7.3)$$

or problems associated with the extraction of principal components of a matrix (so called subspace methods), that is the resolution of

$$\min_{[\hat{H}]} \max_{\{x\}} \frac{\|[H - \hat{H}] \{x\}\|}{\|\{x\}\|} \quad \text{with } \text{rank} [\hat{H}] \ll \text{rank} [H] \quad (7.4)$$

The mathematical tools used to solve such problems are addressed in section A.

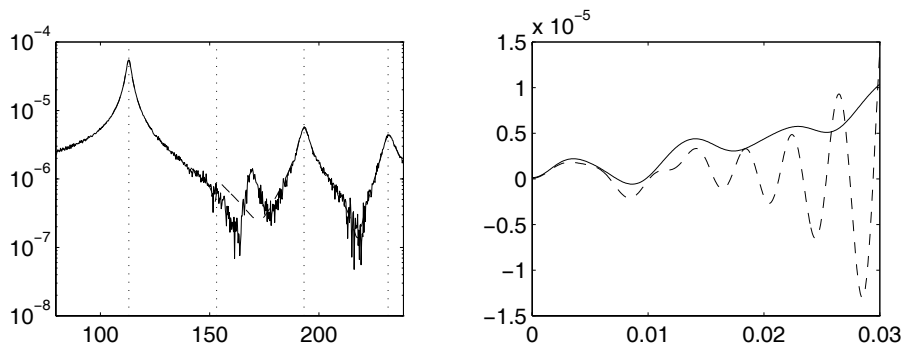
For least-squares problems, one always uses variations on the comparison of measured and predicted quantities. One can for example compare the time response, or the estimated impulse response, ...

$$e = \sum_{k=1}^N \|y_M(k\Delta t) - y_{id}(k\Delta t)\|^2 \quad (7.5)$$

Similarly in the frequency domain, the quadratic norm on estimated transfers in the absence of frequency weighting is given by

$$J = \|H_M(\omega) - H_{id}(\omega)\|^2 \quad (7.6)$$

The quadratic norm in the frequency domain is particularly important since this is what is visualized when overlaying measured and identified transfers. The following simple example, where the second mode is poorly identified, illustrates the accuracy of the diagnostic in the frequency domain (mode 2 is poorly identified), when the comparison of impulse response only shows widely different behavior.



7.2 One mode methods (SDOF)

Avant l'arrivée d'ordinateurs suffisamment puissants, de nombreuses méthodes d'identification à base géométrique furent développées pour l'extraction des paramètres modaux. Bien qu'encore présentes dans les options de nombreux systèmes de mesure, leur intérêt est surtout pédagogique car il permet de bien comprendre les propriétés des modes physiques ce qui peut rendre critique par rapport aux fonctions de transfert mesurées.

Les principaux défauts de ces méthodes sont

- la dépendance forte à la résolution fréquentielle et au savoir faire de l'opérateur
- la difficulté de traiter des systèmes avec plusieurs entrées/sorties
- la quasi-impossibilité de traiter des cas avec modes très proches

Les méthodes à un mode, restent vraiment utiles pour les méthodes d'appropriation, où l'on ajuste les forces appliquées de manière à ce qu'il n'y ai vraiment qu'un seul mode qui réponde (section 8.1.2), et couplées à des algorithmes d'optimisation évolués pour donner une estimation initiale des pôles (section 7.2.2).

7.2.1 Méthodes historiques

On fait l'hypothèse que la réponse est caractérisée par un seul mode (on dit souvent 1 DDL ou *SDOF* mais ce n'est pas très approprié)

$$[H(s)] \approx \frac{A_j}{s^2 + 2\zeta_j\omega_j s + \omega_j^2} \quad (7.7)$$

Le Nyquist de la FRF vitesse/force est alors théoriquement situé sur un cercle.

En étudiant les propriétés de la réponse de l'oscillateur à 1 DDL, on obtient [73, 25], que la fréquence de résonance correspond à la fréquence du maximum de la réponse, le taux d'amortissement est approximativement égal à

$$\zeta_j \approx \frac{\omega_b - \omega_a}{2\omega_n} \quad (7.8)$$

où ω_a et ω_b correspondent à la **demi-puissance** (niveau inférieur de $1/\sqrt{2}$ par rapport au maximum. En supposant le mode isolé est suffisamment amorti pour négliger les termes résiduels, on peut utiliser

$$[H(\omega_j)] \approx \frac{\{c\psi_j\} \{\psi_j^T b\}}{-\zeta_j\omega_j} \quad (7.9)$$

pour estimer la déformée modale.

La méthode de la demi-puissance est vite inefficace si le nombre de points est faible (le pic est alors généralement entre deux fréquences mesurées) ou si les contributions résiduelles des modes voisins sont significatives. La **méthode des cercles** pallie à ce problème en cherchant à faire coïncider un cercle avec les points du diagramme de Nyquist de la réponse au voisinage de la résonance considérée.

Le modèle utilisé est alors de la forme

$$[H(s)] \approx \frac{R_j}{s - \lambda_j} + E \quad (7.10)$$

où R_j et E sont des constantes complexes. ω_j la fréquence naturelle associée au pôle λ_j est estimée en prenant le maximum de pente de la phase ou le point de fréquence ayant la phase la plus proche de celle du centre du cercle. Pour des modes bien séparés, la différence est faible. L'amortissement est estimé à l'aide de

$$\zeta_j \approx \frac{\omega_2 - \omega_1}{\omega_j (\tan(\theta_1/2) + \tan(\theta_2/2))} \quad (7.11)$$

où ω_1 et ω_2 sont des fréquences de part et d'autre de la résonance, θ_1 et θ_2 correspondent aux angles par rapport au rayon du cercle pointant vers le point ayant servi à déterminer ω_j . Le résidu est enfin déterminé par

$$|R_j| = \phi(\text{diamètre}) \text{ et } \tan \alpha = \text{Re}(R_j)/\text{Im}(R_j) \quad (7.12)$$

Ce type de méthode est très rapide mais s'accommode mal de mesures bruitées ou de modes proches en fréquence, demande un opérateur qualifié et donne une estimation différente du pôle pour chaque fonction de transfert.

7.2.2 A modern example

The geometric aspect of the circle method is no longer pertinent in a modern age were numerical libraries to solve least squares problems are easily accessible in MATLAB, Python, ... Taking a simplified model

$$[H(s)] \approx \frac{R_j}{s - \lambda_j} + E = \frac{As + B}{s - \lambda_j} \quad (7.13)$$

and solving it in least squares sense weighted by $s - \lambda_j$

$$\left\{ \begin{array}{c} A \\ B \\ \lambda_j \end{array} \right\} = \arg \min_{[A,B,\lambda_j]} \left| [s \ 1 \ H(s)]_{NW \times 3} \left\{ \begin{array}{c} A \\ B \\ \lambda_j \end{array} \right\} - [sH(s)]_{NW \times 1} \right|^2 \quad (7.14)$$

while considering a restricted set of NW frequencies around a resonance visible in the transfer. The `ii_poest` algorithm, that will be illustrated in lab work, improves this estimate by averaging the poles obtained for each transfer then iteratively resolving the non linear least squares problem

$$\min_{\lambda_j} \min_{R_j, E, F} \left| \left[\frac{1}{s - \lambda_j} \ 1 \ \frac{1}{s^2} \right]_{NW \times 3} \left\{ \begin{array}{c} R_j \\ E \\ F \end{array} \right\}_{3 \times NH} - [H(s)]_{NW \times NH} \right|^2 \quad (7.15)$$

7.3 Méthodes fréquentielles

Le plus souvent, les méthodes d'identification servent à estimer les pôles et éventuellement les commandabilité modales. Les résidus sont obtenus dans un deuxième temps par résolution d'un problème de moindres carrés fréquentiel exposé dans cette section.

7.3.1 Estimation des résidus pour pôles connus

En pratique, tous les algorithmes utiles fournissent une estimation unique des pôles (racines du dénominateur pour les représentations polynômiales, valeurs propres d'un modèle d'état ou d'un modèle du second ordre, pôles d'une décomposition en fraction rationnelle).

Il est alors usuel de faire l'hypothèse d'un modèle diagonal avec termes résiduels choisis en fonction du type de fonction de transfert. On a vu au premier chapitre que ces termes résiduels étaient essentiels pour l'analyse modale.

Dans le cas habituel des fonctions de transfert entre forces et déplacement, on garde ainsi un terme constant pour les contributions des modes haute fréquence (ou en s^2 pour des accélérations mesurées) et, si la structure a des modes en dessous de la bande de fréquence considérée, un terme en $1/s^2$ (constant pour des accélérations) pour représenter leurs effets.

En stockant les différentes fréquences estimées en ligne et chaque paire entrée/sortie en colonne la réponse identifiée peut s'écrire sous la forme

$$[H_k(\omega)]_{NW \times NF} = \left[\left[\frac{1}{i\omega - \lambda_j} \cdots \right] [1] \left[\frac{1}{-\omega_1^2} \right] \right]_{NW \times (NM+2)} \begin{bmatrix} [R_{j,k}] \\ E \\ F \end{bmatrix}_{(NM+2) \times NF} \quad (7.16)$$

où NF est le nombre de fonctions de transfert (typiquement $NF = NS \times NA$) et NM est le nombre de pôles du modèle identifié.

Pour des pôles connus (estimés par une autre méthode), le modèle (7.16) dépend linéairement des résidus et termes résiduels $[R_j(\lambda_j), E, F]$. On peut donc estimer ces termes en résolvant le problème de moindres carrés linéaire associé au coût quadratique dans le domaine fréquentiel

$$[R_j(\lambda_j), E, F] = \arg \min \left| [\alpha(s)]_{test} - \sum_{j=[1, NM]} \left(\frac{[R_j]}{s - \lambda_j} + \frac{[\bar{R}_j]}{s - \bar{\lambda}_j} \right) + [E(s)] \right|^2 \quad (7.17)$$

L'approche est très efficace même pour de très gros problèmes. Cependant

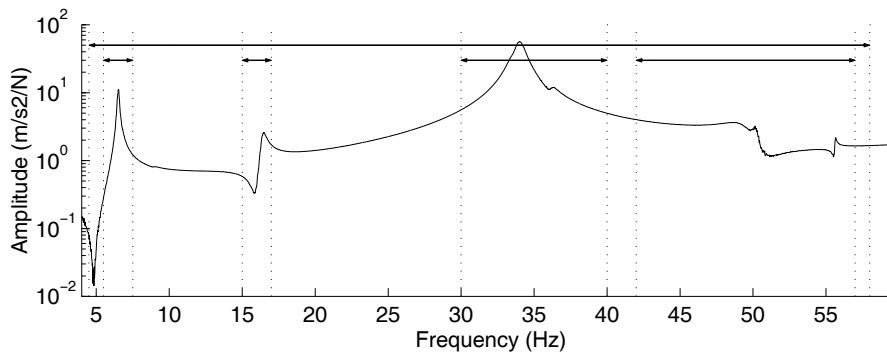
- elle est assez, voire très, sensible à des erreurs sur la position des pôles.
- elle ne prend pas en compte les contraintes de multiplicité (test MIMO), de réciprocité ou de représentation du deuxième ordre (voir section 8.3).

7.3.2 Quelques variantes utiles

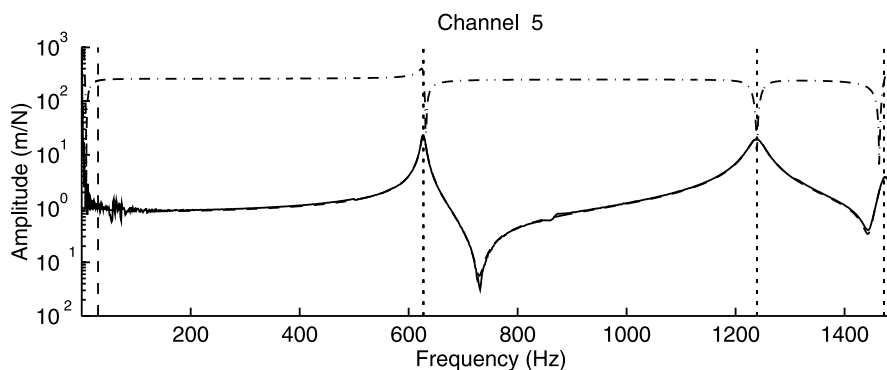
Comme pour tous les problèmes de moindres carrés, il peut être important d'introduire des pondérations sur les espaces d'entrée et de sortie : ici, fréquences et paires actionneur/capteur mesurées.

Pour les entrées, l'utilisation simultanée de capteurs de nature différente (accélérations, vitesses, déformations, ...) conduit souvent à de mauvais résultats. Une solution classique est alors de normaliser chaque fonction de transfert par sa réponse RMS.

La pondération en fréquence est aussi essentielle au bon déroulement de la plupart des identifications. La figure suivante montre ainsi comment on peut identifier les différents pôles en considérant un succession de bandes de fréquences ne contenant que quelques pôles, puis résoudre le problème (7.17) sur l'ensemble de la bande pour obtenir un modèle global cohérent.

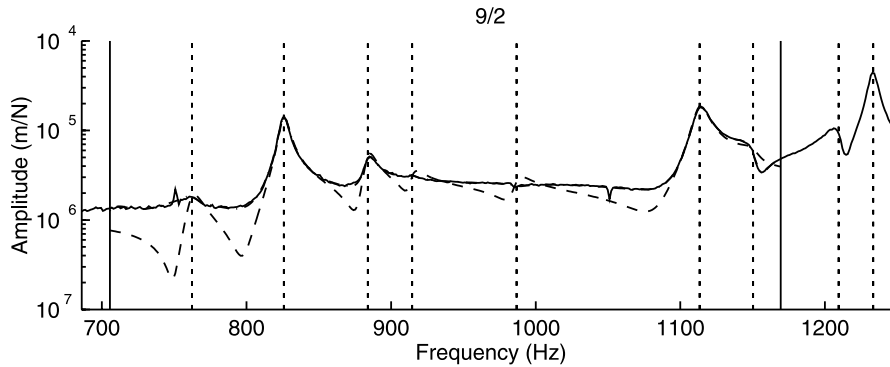


La sélection de bande peut être essentielle, comme le montre l'exemple ci-dessous de mesures allant jusqu'à 0 Hz. Le capteur utilisé étant un accéléromètre piézoélectrique, les données ne correspondent pas à l'accélération en dessous de quelques Hz. La figure montre que des résidus estimés en gardant toutes les fréquences conduisent au résultat idiot (- - -) alors qu'en éliminant les basses fréquences (jusqu'au trait vertical) on obtient une très bonne identification (- - -).



La construction de modèles par bande pose souvent d'autres problèmes liés aux effets des modes voisins. Dans l'exemple suivant, on fait deux identifications dans la bande indiquée par les traits verticaux. Si l'on ne conserve que les modes de la bande (- - -), on voit apparaître des anti-résonances alors que si l'on ajoute les deux modes juste au dessus de la fréquence de coupure, le résultat (- - -) est presque parfait.

Ceci montre qu'il peut être très difficile d'obtenir un résultat correct pour des structures à forte densité modale pour lesquelles, il y a toujours des fréquences un peu au dessus de la bande mesurée.



Des pondérations plus évoluées chercheront à inclure un modèle d'erreur associé aux mesures comme cela est évoqué en [76].

Notez enfin que le modèle pôle/résidu correspond en temporel aux réponses impulsionnelles

$$[h(t)] = \sum_{j=1}^{NM} \left(\{c\psi_j\} \{\psi_j^T b\} e^{\lambda_j t} + \{c\bar{\psi}_j\} \{\bar{\psi}_j^T b\} e^{\bar{\lambda}_j t} \right) = [c\psi_j] [e^{At}] [\psi_j^T b] \quad (7.18)$$

Cette réponse dépend toujours linéairement des résidus, il serait donc possible de déterminer les résidus par un moindres carrés formulé dans le domaine temporel. En pratique, on s'intéresse le plus souvent à des problèmes pour lesquels les fonctions de transfert ou les densités spectrales de puissance sont estimées et utilise donc les données fréquentielles qui, de plus, permettent la troncature parfois indispensable de certaines zones de fréquence.

7.3.3 Moindres carrés non-linéaires

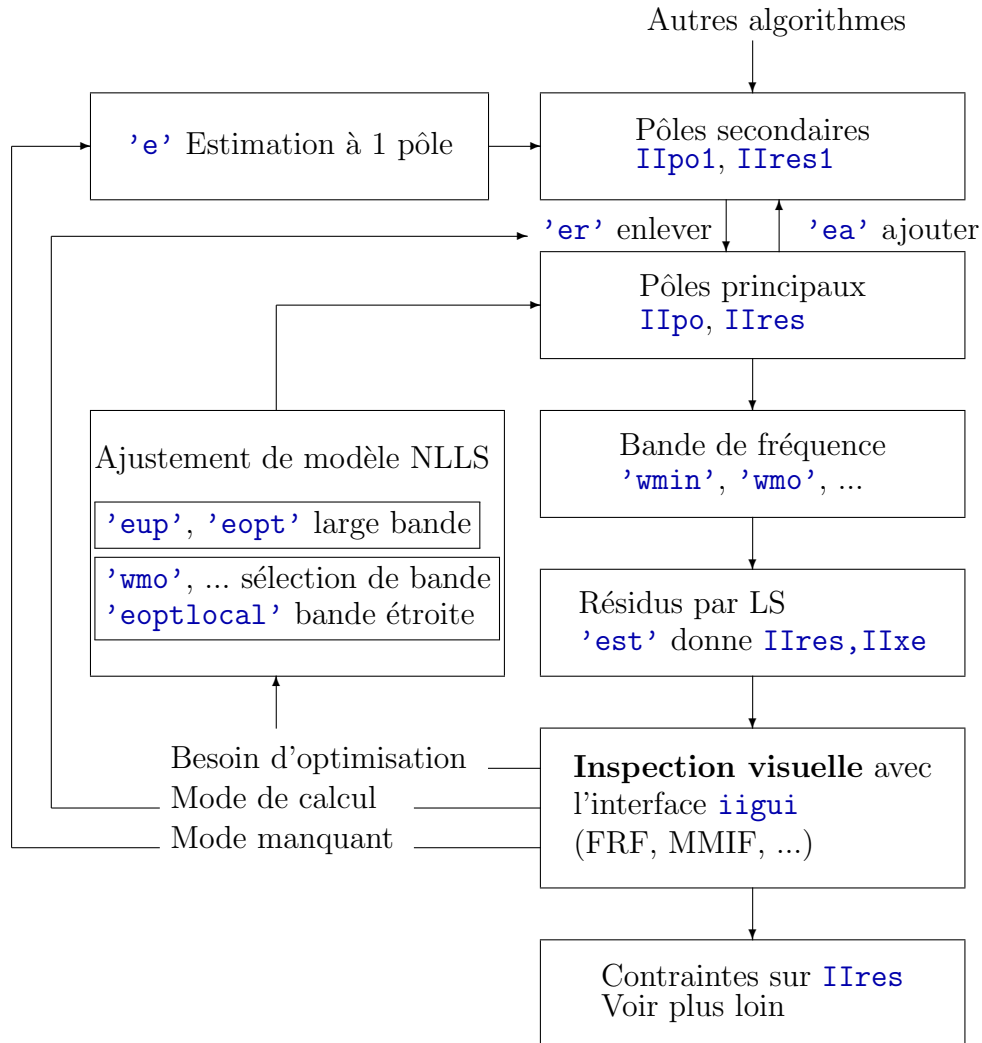
Pour des pôles variables, le problème (7.17) correspond à un problème de moindres carrés non-linéaire (*NLLS-FD : non-linear least squares frequency domain*). La résolution directe du problème non-linéaire est a priori l'algorithme d'identification le plus efficace mais, l'aspect non-linéaire a rendu impossible sa mise en oeuvre pratique avant les années 90. Le nombre élevé de paramètres $(2(NP + 2) \times (NS \times NA))$ qui dépasse très rapidement le millier pour quelques dizaines de pôles et de capteurs et quelques excitateurs) est un aspect essentiel de la difficulté.

L'approche de la *SDT* que vous utiliserez en TP contourne le problème en utilisant deux optimisations imbriquées

$$\min_{\lambda_j} \min_{R_j, E, F} \left| [\alpha(s)]_{test} - \sum_{j \in \text{identified}} \left(\frac{[R_j]}{s - \lambda_j} + \frac{[\bar{R}_j]}{s - \bar{\lambda}_j} \right) + [E(s)] \right|^2 \quad (7.19)$$

Les résidus sont donc considérés comme des fonctions implicites des pôles obtenues par résolution du problème de moindres carrés linéaire (7.17). On utilise alors le gradient du coût pour un changement des pôles pour construire un méthode d'optimisation des pôles assez efficace [77].

Le fonctionnement général de la procédure d'identification par optimisation du modèle est résumé dans le schéma ci-dessous. Cette approche bien qu'éprouvée sur de nombreux problèmes industriels pourrait faire l'objet de nombreuses extensions.



7.4 Rational fraction representations

Polynomial and rational fraction representations have long been considered in control theory. A general form of such problems is [28]

$$A(q)y(t) = \frac{B(q)}{F(q)}u(t) + \frac{C(q)}{G(q)}e(t) \quad (7.20)$$

where $A(q)y(t)$ corresponds to the discrete time model $a_0y(t) + a_1y(t - \delta t) + \dots + a_{n_a}y(t - n_a\delta t)$. The most common time domain algorithms are ARX (polynômes A et B), ARMA (A et C), et OE (output error, B et F).

These methods are known as **system identification** see [28] and the *System Identification Toolbox* of MATLAB. They are a number of limitations for their use in vibration problems where the number of poles of sensors is usually quite large.

The polynomial or rational fraction form can also be used in the frequency domain form as

will be illustrated later. Taking the form

$$H_j(s) = \frac{a_{j,1}s^{na-1} + a_{j,2}s^{na-2} + \dots + a_{j,na}}{b_{j,1}s^{nb-1} + b_{j,2}s^{nb-2} + \dots + b_{j,nb}} \quad (7.21)$$

accounting for constraints that poles do not depend on each transfer or that modeshapes are fully described by their observability and commandability can be difficult. [74] thus provides a classification of polynomial methods where matrix polynomials are considered.

The current state of the art is the LSCF method, also known under the commercial name Polymax, see [78, 79] or in French [75].

7.4.1 First example : frequency domain rational fraction

For a single transfer, one seeks a model of the form

$$H(\{a\}, \{b\}, s) = \frac{a_0 + a_1s + \dots + a_ns^n}{1 + b_1s + \dots + b_ds^d} \quad (7.22)$$

One would ideally want to minimize the output error in the frequency domain

$$J = \left\| \frac{A(s)}{B(s)} - H_M(s) \right\|^2 \quad (7.23)$$

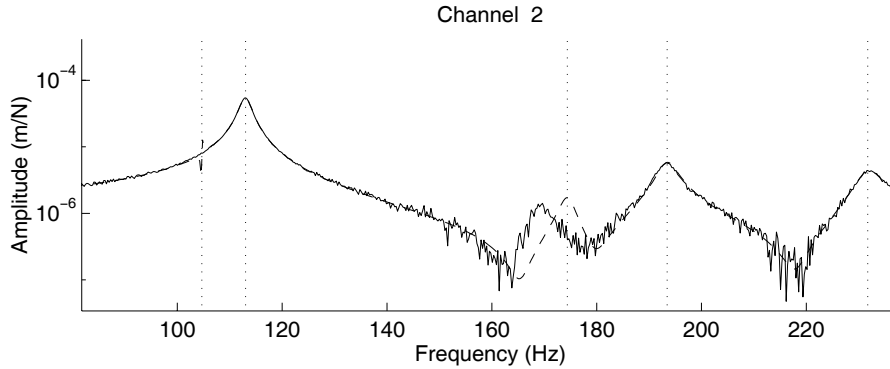
but this leads to a non-linear optimization problem. One thus prefers a modified objective function given by

$$J = \|A(s) - B(s)H_M(s)\|^2 \quad (7.24)$$

Assuming the transfer function to be measured at frequency points $\{i\omega_1, \dots, i\omega_w\}$, the minimum of J is solution of the least squares problem minimizing the residual given by

$$\{R\} = \begin{bmatrix} 1 & \dots & (i\omega_1)^n & \dots & (i\omega_1)^d H_M(\omega_1) \\ \vdots & & \vdots & & \vdots \\ 1 & \dots & (i\omega_w)^n & \dots & (i\omega_w)^d H_M(\omega_w) \end{bmatrix} \begin{Bmatrix} a_0 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_d \end{Bmatrix} - \begin{bmatrix} H_M(\omega_1) \\ \vdots \\ H_M(\omega_w) \end{bmatrix} \quad (7.25)$$

The following example (command `idcom('poly 10 10')` on the `demo_id` data of SDT), illustrates fairly well the problems with such methods. One sees a **computational mode** close to 105 Hz (a non physical pole allowing a better fit) and the shift of the second mode (this mode having a low contribution, the fit seeks a better fit of noise at high amplitudes rather than improving this resonance). Weighting could correct these problems, but an automated weighting strategy is not trivial.



Quelques unes des difficultés additionnelles sont

- le conditionnement numérique pour des polynômes d'ordre élevé est très mauvais. Ceci peut être résolu avec la méthode des polynômes orthogonaux [80] ou en considérant des fréquences normalisées (transformée en Z par exemple).
- Il est difficile d'imposer l'unicité des pôles. Quelle que soit la paire entrée/sortie ij on voudrait imposer que les pôles soient commun

$$H_{i,j}(\{a\}, \{b\}, s) = \frac{A_0(i, j) + A_1(i, j)s + \dots + A_n(i, j)s^n}{1 + b_1s + \dots + b_d s^d} \quad (7.26)$$

on peut définir un problème de moindres carrés associé mais le coût numérique croît de manière rapide (matrice de très grande taille). Mais des simplifications sont possibles [79]

- il est difficile de choisir l'ordre des polynômes. On utilise les diagrammes de stabilisation pour cela.
- si le résultat est mauvais, on ne sait pas bien comment changer les paramètres du calcul : ordre des polynômes, bande de fréquence, pondération
- en pratique le meilleur coût est donné par

$$J = \|A(s)/B(s) - H_M(s)\|^2$$

qui est un problème de moindres carrés non linéaire en B.

- comme toujours, il est difficile d'obtenir un modèle global (imposer l'unicité des vecteurs propres, ..., voir section 8.3).

7.4.2 Polyreference Least Squares Complex Exponential (LSCE)

L'utilisation d'un modèle d'état n'est pas très efficace dans une approche de résolution par moindres carrés. L'algorithme LSCE, qui sert généralement de référence dans les applications d'analyse modale expérimentale, utilise donc un polynôme à coefficients matriciels de dimension $NA \times NA$ (nombre d'entrées).

La réponse impulsionnelle d'un modèle pôle/résidu sans termes résiduels est de la forme

$$[h(n\Delta t)]_{NS \times NA} = [c\psi] \begin{bmatrix} e^{\Lambda n\Delta t} \\ \vdots \end{bmatrix} [\psi^T b] \quad (7.27)$$

Pour un système à $2NM$ pôles, il existe un polynôme caractéristique d'ordre $p = 2NM/NA$ à coefficients matriciels W_p de dimension $NA \times NA$ tel que

$$\left[\begin{array}{c} \setminus z_j^p \\ \setminus \end{array} \right] \{ \psi_j^T b \} \left[\setminus I \setminus \right] + \left[\begin{array}{c} \setminus z_j^{(p-1)} \\ \setminus \end{array} \right] \{ \psi_j^T b \} [W_1] + \dots + \left[\begin{array}{c} \setminus z_j^{(0)} \\ \setminus \end{array} \right] \{ \psi_j^T b \} [W_p] = \{0\} \quad (7.28)$$

avec $z_j = e^{\lambda_j \Delta t}$.

L'équation (7.27) montre que les réponses impulsionnelles sont des combinaisons linéaires des solutions $z_j \{ \psi_j^T b \}$ du polynôme caractéristique. On détermine donc les coefficients $[W]$ en résolvant au sens des moindres carrés l'équation

$$[h(n\Delta t)] \left[\setminus I \setminus \right] + [h((n-1)\Delta t)] [W_1] + \dots + [h((n-p)\Delta t)] [W_p] = \{0\} \quad (7.29)$$

Ayant déterminé les $[W]$, on résout (7.28) pour déterminer les pôles et facteurs de participation, puis revient au problème de moindres carrés linéaire (7.17) pour déterminer des déformées $c\psi_j$.

7.4.3 Polynômes matriciels du second ordre

De nombreux auteurs ont introduits des méthodes basées sur l'utilisation de polynômes à dénominateur matriciel d'ordre 2 [81, 82, 83]. La forme de base considérée est

$$\begin{aligned} [-\omega^2 I + i\omega C_T + K_T] \{p(\omega)\} &= [b_T] \{u(\omega)\} \\ \{y(\omega)\} &= [c_T] \{p(\omega)\} \end{aligned} \quad (7.30)$$

qui fait clairement apparaître que pour $[c_T]$, $\{y_T\}$, $\{u_T\}$ connus, les matrices du modèle $[C_T]$, $[K_T]$, et $[b_T]$ peuvent être obtenues par résolution d'un problème de moindres carrés linéaire.

Le contenu fréquentiel de l'entrée $\{u\}$ a une forte influence sur le résultat obtenu et bien souvent il est nécessaire de l'utiliser comme pondération plutôt que de prendre une entrée blanche (valeur unitaire à toutes les fréquences).

Les grandes difficultés de ce type de modèle sont comme toujours, la prise en compte des termes résiduels, de la réciprocity, ou des conditions de positivité des différentes matrices. Bien que justifiée par une base mécanique (le modèle de référence est du second ordre), ces algorithmes sont souvent moins performants que d'autres.

7.5 Subspace methods

Subspace methods are traced back to work by Ho and Kalman [84]. More recent books are [85, 86]. The underlying principle is that the response of a system with a finite number of states lies within a finite subspace. One thus considers discrete state space models of the form

$$\begin{aligned} \{x((k+1)\Delta t)\} &= [A] \{x(k\Delta t)\} + [B] \{u(k\Delta t)\} \\ \{y(k\Delta t)\} &= [C] \{x(k\Delta t)\} \end{aligned} \quad (7.31)$$

7.5.1 Ibrahim Time Domain Method

Pour un modèle sans termes résiduels, on montre que la matrice de Hankel définie par

$$[h(t)_{ik}] = \begin{bmatrix} [h(t)] & [h(t + \Delta t)] & \dots & [h(t) + (k-1)\Delta t] \\ [h(t + \Delta t)] & [h(t + 2\Delta t)] & \dots & [h(t) + (k)\Delta t] \\ \vdots & \vdots & \ddots & \vdots \\ [h(t) + (i-1)\Delta t] & [h(t) + (i)\Delta t] & \dots & [h(t) + (k+i-2)\Delta t] \end{bmatrix} \quad (7.32)$$

peut s'écrire sous la forme $[h(t)_{ik}] = [V] [e^{\Lambda(k-1)\Delta t}] [L]$

$$[h(t)_{ik}] = \begin{bmatrix} [c\psi] \\ \vdots \\ [c\psi] [e^{\Lambda(i-1)\Delta t}] \end{bmatrix} [e^{\Lambda t}] [[\psi^T b] \dots [e^{\Lambda(k-1)\Delta t}] [\psi^T b]] \quad (7.33)$$

On définit $[W]$ comme solution du problème de moindres carrés

$$[h(t + \Delta t)]_{ik} = [W] [h(t)]_{ik} \quad (7.34)$$

En utilisant l'hypothèse d'un modèle sans terme résiduels et donc la forme particulière (7.33) de la matrice de Hankel, on vérifie que W peut s'écrire sous la forme

$$[W] = [V] [e^{\Lambda\Delta t}] [V]^+ \quad (7.35)$$

où $[V]^+$ est un pseudo-inverse de $[V]$. Ayant trouvé W , on peut donc trouver les pôles et modes en résolvant le problème aux valeurs propres

$$[W] [V] = [V] [e^{\Lambda\Delta t}] \quad (7.36)$$

Cette méthode correspond à l'utilisation d'un modèle d'état discret (polynôme d'ordre 1 à coefficients matriciels) de la forme

$$\{y(t + \Delta t)\} = [W] \{y(t)\} + [B] \{u(t)\} \quad (7.37)$$

où l'on considère des réponses impulsionnelles donc $\{u(t)\} = 0$.

Dans cette méthode, le nombre l'ordre du modèle dépend directement du nombre de décalages temporels considérés (taille de la matrice de Hankel). Ceci rend l'algorithme très sensible à toute erreur sur les premiers points de la réponse impulsionnelle. Il est donc préférable d'utiliser les méthodes de sous-espace comme l'algorithme ERA abordé en section 7.5. Ces méthodes utilisent la SVD plutôt que la résolution d'un problème de moindres carrés pour obtenir un modèle d'ordre donné.

7.5.2 Input-output illustration : ERA

As a first illustration, one can consider the ERA (*Eigensystem Realisation Algorithm*) which assumes zero initial conditions and an impulse response (i.e. $[u(k\delta t)] = [I] \delta_{k0}$). Responses at various time steps form the so called Markov parameters

$$\{y(k\Delta t)\} = [C] [A^k] [B] \quad (7.38)$$

that one can assemble as a generalized Hankel matrix

$$[h(k\Delta t)]_{pq} = \begin{bmatrix} [y(k)] & [y(k + j_1)] & \dots & [y(k + j_{q-1})] \\ [y(k + i_1)] & [y(k + i_1 + j_1)] & \dots & [y(k + i_1 + j_{q-1})] \\ \vdots & \vdots & \ddots & \vdots \\ [y(k + i_{p-1})] & [y(k + i_{p-1} + j_1)] & \dots & [y(k + i_{p-1} + j_{q-1})] \end{bmatrix} \quad (7.39)$$

One can easily show that the Hankel matrix can be written in the form

$$[h(k\Delta t)]_{pq} = [C]_p [A]^{k-1} [B]_q = \begin{bmatrix} [C] \\ \vdots \\ [C] [A]^{j_{p-1}} \end{bmatrix} [A]^{k-1} \begin{bmatrix} [B] & \dots & [A]^{j_{q-1}} [B] \end{bmatrix} \quad (7.40)$$

where $[C]_p$ is called the generalized observation matrix and $[B]_q$ is the generalized commandability matrix.

There thus exists a rank constraint on the true Hankel matrix (the rank of this matrix is equal to the number of modes in the model). One thus estimates Markov parameters experimentally, then uses the SVD to build a rank constrained approximation of the Hankel matrix

$$[h(k\Delta t)]_{pq} \approx [U] [\backslash S \backslash] [V]^T \quad (7.41)$$

Finally, there are standard expressions giving a state-space realization that has an Hankel matrix given by $[U] [\backslash S \backslash] [V]^T$. For the ERA algorithm, one for example uses

$$\begin{aligned} [A] &= \left[[\backslash S \backslash]^{-1/2} [U]^T [h(2\Delta t)]_{pq} [V] [\backslash S \backslash]^{-1/2} \right] \\ [B] &= [\backslash S \backslash]^{1/2} [V]^T \begin{bmatrix} [\backslash I \backslash] \\ [0] \\ [0] \end{bmatrix}_{qNA \times NA} \\ [C] &= \left[[\backslash I \backslash] [0] [0] \right]_{NS \times pNS} [U] [\backslash S \backslash]^{1/2} \end{aligned} \quad (7.42)$$

7.5.3 Output only methods

A major interest of subspace methods is that they are also applicable to output only measurements. Typical applications are monitoring of civil engineering structures and flight measurements of aircraft and spacecraft.

In such applications, only the outputs $y(t_k)$ are known. One then considers an Hankel matrix associated with covariances $R_i = E(Y_k Y_{k-i}^T)$

$$[h(k\Delta t)]_{pq} = \begin{bmatrix} [R(0)] & [R(1)] & \dots & [R(q-1)] \\ [R(1)] & [R(2)] & \dots & [R(1+q-1)] \\ \vdots & \vdots & \ddots & \vdots \\ [R(p-1)] & [R(p-1+1)] & \dots & [R(p-1+q-1)] \end{bmatrix} \quad (7.43)$$

introducing the covariance between states and outputs $G = E(X_k Y_k^T)$ direct computation of the output covariances from the discrete state-space equation (7.31) yields $R_i = CA^i G$ and a classical factorization of the Hankel matrix of the form

$$[h(k\Delta t)]_{pq} = [O_p(C, A)] [C_q(A, G)] \quad (7.44)$$

where

$$[O_p(C, A)] = \begin{bmatrix} [C] \\ \vdots \\ [C] [A]^{p-1} \end{bmatrix} \quad \text{and} \quad [C_q(A, G)] = [G \ AG \ \dots \ A^{q-1}G] \quad (7.45)$$

are the observability and controllability matrices respectively. The observation matrix C is then found in the first block-row of the observability matrix O . The state transition matrix A is obtained from the shift invariance property of O namely

$$[O_p^+(C, A)] = [O_p(C, A)] A, \quad \text{where} \quad [O_p^+(C, A)] = \begin{bmatrix} [C] A \\ \vdots \\ [C] [A]^p \end{bmatrix} \quad (7.46)$$

where it is clearly assumed that the rank of O_p is equal to the size of A and that the number of block rows in H_{pq} to recover A . This expression is simply associated with a linear least squares problem from which A can be estimated. Knowing A , poles and modeshapes can be computed and observed using the estimated C .

The estimation of observabilities and modeshape scaling has been the object of various studies.

Chapitre 8

Practical tools used for identification

En pratique, le processus d'identification doit permettre d'extraire une caractérisation du comportement dynamique et d'obtenir une idée qualitative de sa validité. Il est donc indispensable de combiner des algorithmes numériques proposant des solutions raisonnables et un opérateur jugeant de leur validité. Les grandes étapes sont

- estimer les pôles
- sélectionner les pôles physiques
- estimer les résidus et valider le résultat
- prendre en compte des hypothèses de réciprocité, multiplicité, existence d'un modèle élastique sous-jacent.

Les principaux outils disponibles sont liés

- au choix du rang/nombre de pôles (SVD, MMIF, diagramme de stabilisation et d'erreur, ... abordés en section 8.1)
- à la sélection des pôles physiques (critères basés sur la superposition de FRFs, visualisation de déformées, ... abordés en section 8.2)
- au raffinement des résultats initiaux (sélection de bandes de fréquence, pondérations, moindres carrés non-linéaires, prise en compte d'hypothèses supplémentaires, ... voir section 8.3)

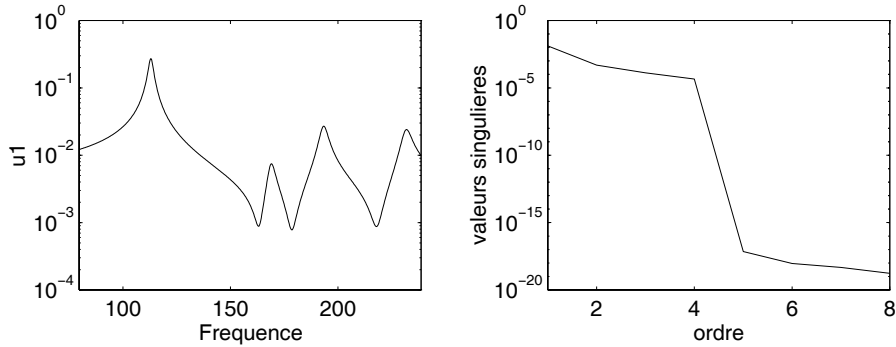
8.1 Détermination de l'ordre

8.1.1 SVD de la matrice de FRF

Pour un modèle sans termes résiduels, on peut écrire les fonctions de transfert sous la forme d'un produit d'une matrice de contributions modales par une matrice de résidus

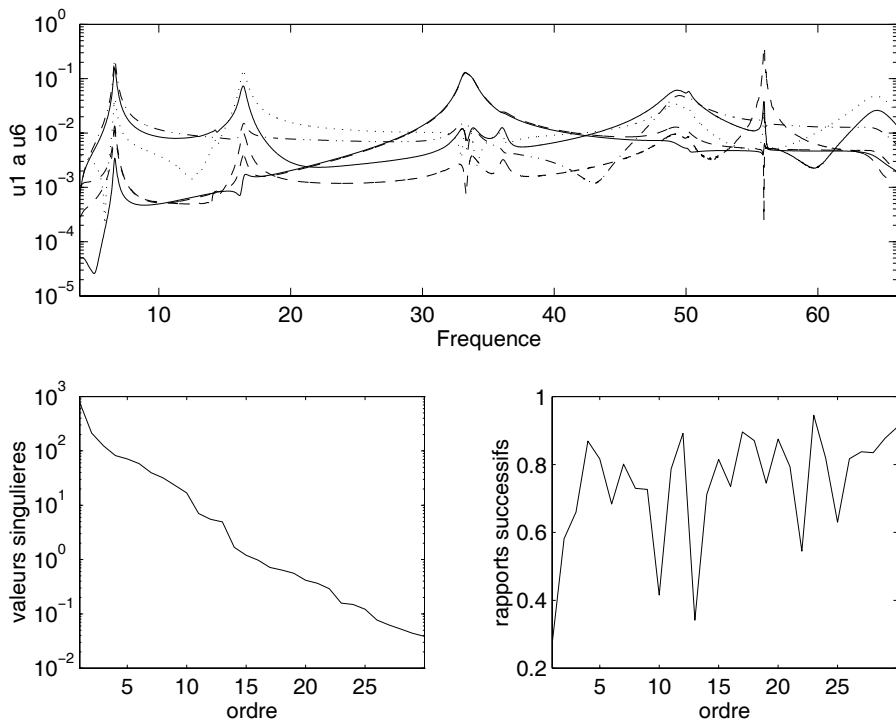
$$[H_{kl}(\omega)]_{NW \times N_{kl}} = \left[\frac{1}{\omega_{1:NW} - \lambda_{1:NM}} \right]_{NW \times NM} [R_{j,kl}]_{NM \times N_{kl}} \quad (8.1)$$

On voit donc que la décomposition en valeurs singulière de la matrice des fonctions de transfert (avec les lignes correspondant aux fréquences et colonnes aux paires entrée/sortie) doit donner au plus NM valeurs singulières significatives. Sur le cas théorique ci-dessous, cela marche très bien. Il y a 4 modes et 4 valeurs singulières significatives.



Dans le cas réel, on peut tracer les vecteurs singuliers à gauche qui permettent effectivement de mettre en évidence les pics les plus significatifs et peuvent être utilisés pour l'identification.

Par contre le tracé des valeurs singulières donne une décroissance régulière. Pour détecter les sauts, le tracé du rapport des valeurs singulières successives est plus utile. On voit ici que l'ordre 10 semble raisonnable. Or dans le cas présent, il faut pour la bande considérée utiliser 8 modes et 2 termes résiduels, l'ordre 10 est donc bien une information utile. Ce cas est cependant plutôt une exception et il est souvent difficile de détecter l'ordre à utiliser à partir des sauts.



8.1.2 Fonctions indicatrices de mode (MIF) et appropriation

Le MMIF (*multivariate mode indicator function*) introduit en [87] est utilisé pour détecter le nombre de pôles d'une structure faiblement amortie. On suppose que les données représentent des fonctions de transfert entre force et déplacement, vitesse ou accélération.

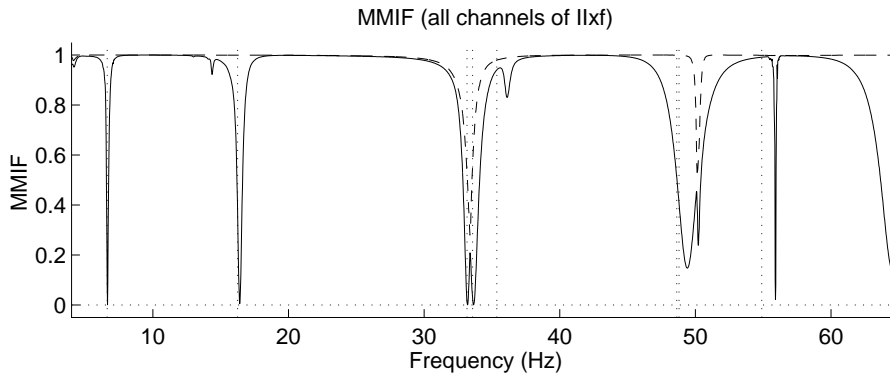
A la résonance, la réponse d'un mode isolé est purement imaginaire (voir l'oscillateur à 1 DDL, voir section 1.1). Le MMIF détecte les résonances en regardant le rapport d'amplitude entre réponse imaginaire et réponse totale (pour une mesure en vitesse)

$$q(s) = \min_{\{u\} \neq 0} \frac{\text{Im}\{y\}^T \text{Im}\{y\}}{\{y\}^H \{y\}} = \min_{\{u\} \neq 0} \frac{\{u\}^T \left[\text{Im}[H]^T \text{Im}[H] \right] \{u\}}{\{u\}^T \text{Re} \left[[H]^H [H] \right] \{u\}} \quad (8.2)$$

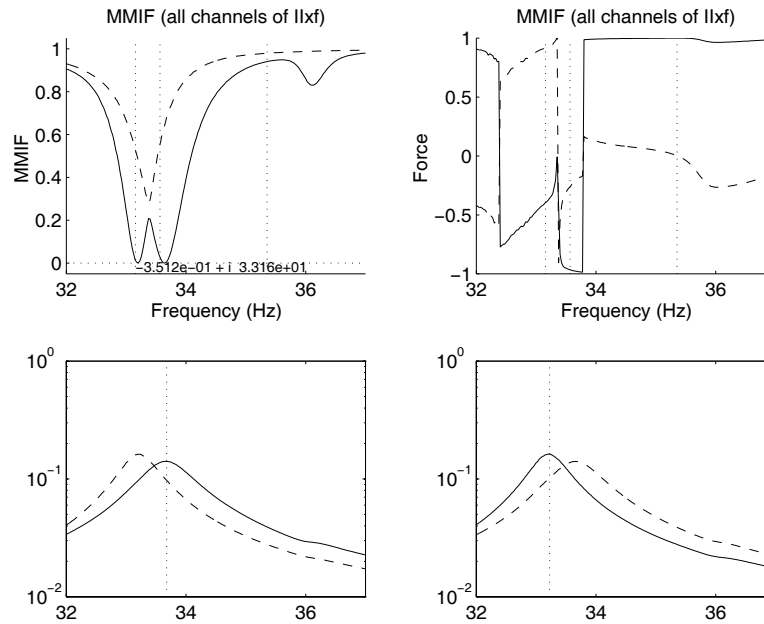
Il existe de nombreuses variantes du MMIF adaptées pour des mesures en vitesse, accélération ou déplacement, ... [88]. Pour un système à plusieurs entrées, le MMIF est associé au problème aux valeurs propres

$$\left[\text{Im}[H]^T \text{Im}[H] \right] \{u_j(s)\} q_j(s) = \text{Re} \left[[H]^H [H] \right] \{u_j(s)\} \quad (8.3)$$

On résout et trace l'évolution des q_j en fonction de la fréquence. Les minima indiquent la présence de modes. La présence de minima doubles (minimum de q_1 et q_2) indiquent la présence de modes multiples. Pour un test à 2 entrées sur la maquette GARTEUR, on obtient le tracé ci-dessous où l'on voit clairement la présence de deux modes au voisinage de 33 Hz.



Dans le cas à plusieurs entrées, les forces $\{u_j\}$ associées à un minimum du MIF permettent d'isoler la réponse d'un mode unique (rendre presque inobservable tous les modes sauf celui contribuant le plus à rendre la réponse presque imaginaire). Pour le cas idéal où il y aurait autant d'entrées que de modes, la force ainsi déterminée conduirait à une réponse avec un seul mode. En pratique, on obtient des résultats comme celui ci-dessous, où l'application des forces conduit à des FRFs avec une réponse faible des modes voisins.

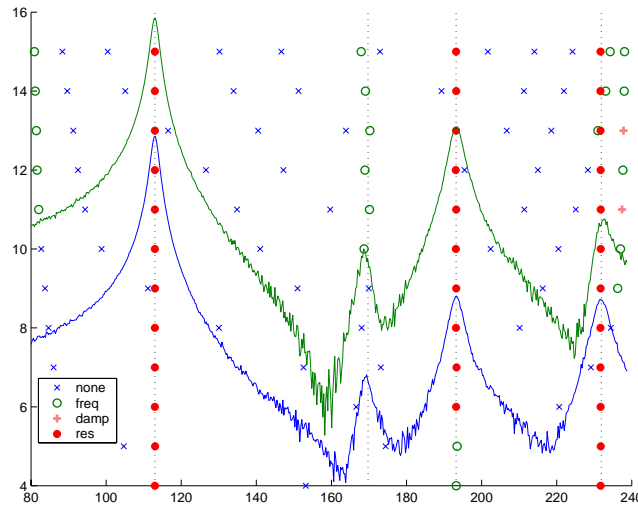


Les méthodes d'**appropriation modale** [88] cherchent déterminer des combinaisons linéaires forces (et les endroits où les appliquer) rendant un seul mode vraiment observable. C'est une procédure souvent longue, mais, une fois les forces déterminées, on peut facilement mesurer la déformée en un très grand nombre de capteurs et l'identification ne pose aucun problème (une méthode à 1 pôle avec termes résiduels suffit).

Le CMIF (*Complex Mode Indicator Function*, [89]) est un critère alternatif au MMIF. Il utilise une décomposition en valeur singulière de $H(s)$ à chaque fréquence, où les résonances apparaissent clairement comme des maxima de la plus grande valeur singulière. Ce critère ne fait pas référence aux propriétés élastiques et est donc d'utilisation plus générale.

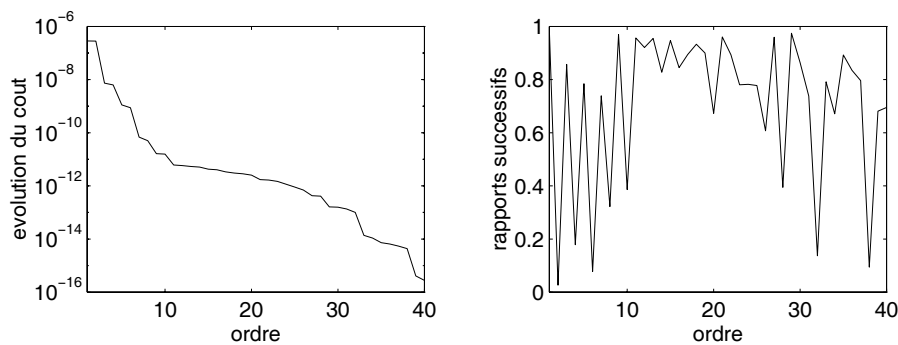
8.1.3 Diagramme de stabilisation et d'erreur

Le paramètre le plus aisément modifiable de nombreuses méthodes est l'ordre (nombre de pôles) du modèle. Il est courant et généralement peu coûteux en temps d'estimer le modèle pour une gamme significative d'ordres. Pour chaque ordre, on cherche les pôles de l'ordre précédent voisins en fréquence, amortissement, ou résidu associé. En pratique, on cherchera par exemple des pôles à moins de 0.5% en fréquence, 5% en amortissement, 10% en norme du résidu. Le tracé de ce type de recherche par ordre croissant en faisant ressortir les propriétés se stabilisant quel que soit l'ordre est appelé **diagramme de stabilisation** et ressemble à celui montré ci-dessous.



En pratique, les modes stables (en fréquence/amortissement/résidu) correspondent à des modes physiques. L'opérateur peut donc sélectionner de tels modes avec la souris. La stabilisation étant déterminée par rapport à des seuils de proximité, il est clair que le processus est assez dépendant de la l'expertise de l'opérateur. Par ailleurs, les pôles qui ne se stabilisent pas ne sont pas forcément non-physiques, ...

En parallèle du diagramme de stabilisation, on trace souvent l'évolution du coût en fonction de l'ordre. S'il y a décrochement significatif, on peut supposer que l'ordre réel est atteint. En pratique cependant, les bruits de mesure et les contributions résiduelles des modes hors de la bande testée conduisent à des diagrammes d'erreur sans décrochement et donc peu utiles.



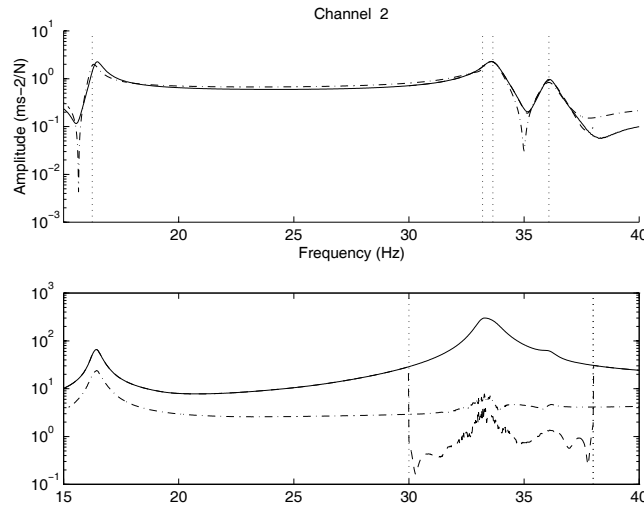
8.2 Validation du résultat

8.2.1 Comparaison des FRF mesurées/synthétisées

Les diagrammes de stabilisation, comme l'approche de raffinement itératif de modèle, conduisent à une estimation des pôles, il est indispensable de juger ensuite de la validité de chaque mode et du modèle d'ensemble.

L'approche standard est la superposition graphique des FRFs mesurées sur les diagrammes de Bode (comme ci-dessous) ou de Nyquist. On peut aussi comparer l'amplitude relative de H_{Test} et de $H_{Test} - H_{Id}$. Cette approche permet dans l'exemple ci-dessous de montrer que le

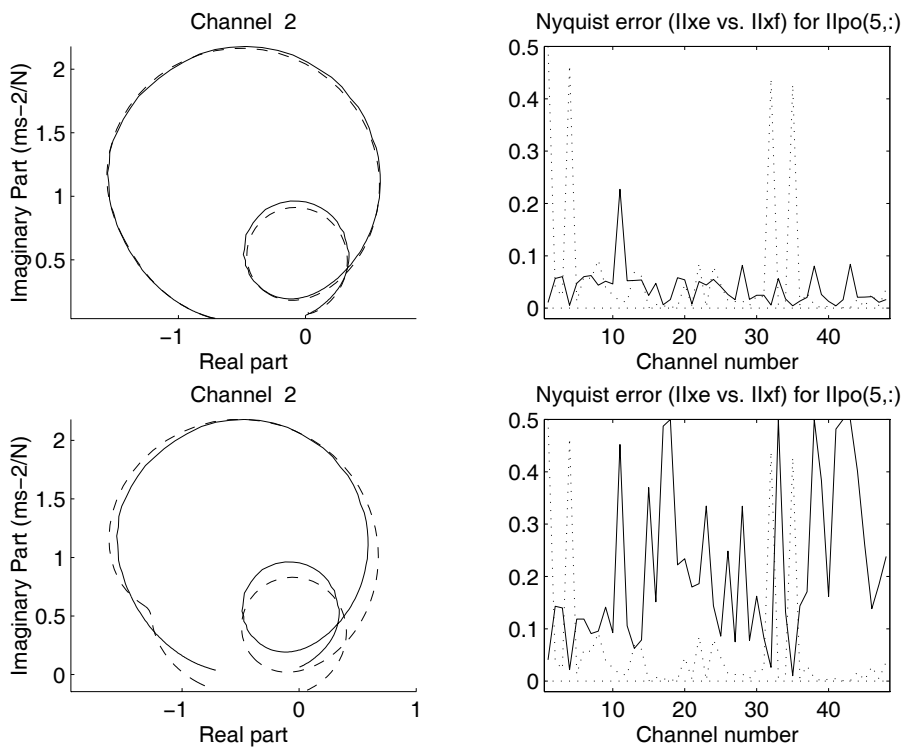
niveau d'erreur pour une identification dans la bande 30-38 Hz est significativement plus faible si l'on conserve les modes hors de la bande.



Pour le diagramme de Nyquist, il est utile de tracer la réponse au voisinage du pic (soit dans la bande $[\omega_j(1 - \zeta_j) \ \omega_j(1 + \zeta_j)]$). On peut alors calculer pour chaque fonction de transfert une erreur sur cette bande de fréquence

$$e_j = \frac{\int_{\omega_j(1-\zeta_j)}^{\omega_j(1+\zeta_j)} |H_{Test} - H_{Id}|^2}{\int_{\omega_j(1-\zeta_j)}^{\omega_j(1+\zeta_j)} |H_{Test}|^2} \tag{8.4}$$

Le tracé de cette erreur sur l'ensemble des voies permet d'avoir une bonne idée des fonctions de transfert pour lesquelles le résidu est bien estimé.

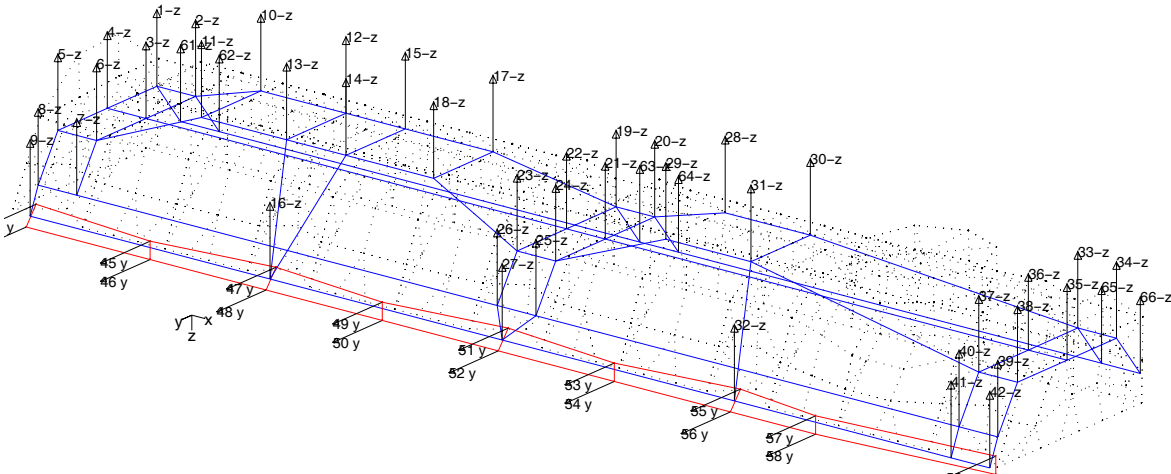


8.2.2 Visualisation des déformées

La visualisation des déformées est un aspect essentiel de la validation d'un résultat d'identification. L'irrégularité du résultat permet en effet souvent de détecter un problème d'identification ou de mesure. Les grandes étapes de ce travail sont

- le positionnement 3-D des points mesurés. C'est l'étape la plus longue en termes de main d'oeuvre, sauf utilisation de systèmes électromagnétiques ou acoustiques de triangulation permettant la mesure immédiate de la position d'un point quelconque.
- la création d'un maillage de visualisation. Ce maillage ne correspond pas à un modèle physique particulier mais permet à l'opérateur de juger de la pertinence physique du modèle considéré. Dans les cas 3-D avec des capteurs non triaxiaux, il est facile de donner une mauvaise impression du mouvement réel. Cette étape est donc très importante.
- la création d'interpolations permettant d'estimer le mouvement en des points ou dans des directions non-mesurées (section 9.5).

Pour illustrer ce processus considérons l'exemple suivant d'un essai au vibromètre sur un couvre culasse Renault. Cette essai comprend 48 mesures verticales et 18 horizontales. Les mesures horizontales et verticales étant en des points différents, il convient de séparer le maillage de visualisation en 2 pour chaque type de mouvement. L'animation de chaque sous maillage pourra se faire séparément pour permettre une bonne analyse du mouvement dans chaque direction.



Le maillage élément fini, visualisé ici en pointillé, n'est pas coïncident avec le maillage expérimental. Une corrélation calculs/essais efficace passe donc nécessairement par une interpolation permettant de connaître le mouvement élément fini aux noeuds expérimentaux. Ce problème est abordé plus en détail en section 9.5.

8.2.3 Propriétés des résidus

Une littérature fournie existe sur la validation des résultats d'identification à partir de l'analyse des propriétés des résidus. On ne citera ici que quelques méthodes significatives.

Pour des identifications utilisant une forme d'état faisant intervenir déplacements et vitesses, on obtient deux estimations de la déformée associées au déplacement et à la vitesse. La cohérence de ces deux estimations peut alors être utilisée comme indicateur de la validité du résultat (*Modal Confidence Factor* [90]).

Les systèmes identifiés étant mécaniques, on s'attend à ce que l'ajout d'une masse ponctuelle abaisse les fréquences de tous les modes (pour une masse distribuée qui modifie aussi la raideur, les fréquences peuvent augmenter). La sensibilité de la fréquence à une masse ajoutée est donnée par

$$\frac{\partial \text{Im} \lambda_j}{\partial M_m} = \text{Im} (-\lambda_j^2 c_m \psi_j \psi_j^T b_m) \quad (8.5)$$

Les résidus collocalisés $R_{m,j} = c_m \psi_j \psi_j^T b_m$ des modes physiques devraient donc vérifier $\text{Im} \lambda_j^2 R_{m,j} > 0$. La non-vérification de ce critère donne une indication de la qualité de l'identification.

Pour une structure faiblement amortie, les modes complexes sont censés être assez proches de modes normaux et donc avoir des coefficients dont la phase diffère de 0° ou 180° . Pour un système réciproque et la normalisation conduisant à (8.18), les observabilité et commandabilité modales sont censées être sur une ligne à -45° . Cependant certaines erreurs d'identification peuvent conduire à un résultat aligné sur une autre phase.

Le critère MPC (*Modal Phase Collinearity*) évalue le degré de dispersion de la phase d'un résidu. Pour calculer ce critère, on commence par recentrer le vecteur

$$\{\tilde{\psi}_j\} = \{\psi_j\} - \{\psi_j\}^T \begin{Bmatrix} 1 \\ \vdots \\ 1 \end{Bmatrix} / NS \quad (8.6)$$

puis à l'aide des quantités

$$\epsilon = \frac{|\text{Im} \{\tilde{\psi}_j\}|^2 - |\text{Re} \{\tilde{\psi}_j\}|^2}{2 \text{Re} \{\tilde{\psi}_j\}^T \text{Im} \{\tilde{\psi}_j\}} \quad \text{et} \quad \theta = \arctan (|\epsilon| + \text{sign}(\epsilon) \sqrt{1 + \epsilon^2}) \quad (8.7)$$

on calcule le critère

$$MPC_j = \frac{|\text{Re} \{\tilde{\psi}_j\}|^2 + \left(\text{Re} \{\tilde{\psi}_j\}^T \text{Im} \{\tilde{\psi}_j\} \right) (2(\epsilon^2 + 1) \sin^2 \theta - 1) / \epsilon}{|\text{Im} \{\tilde{\psi}_j\}|^2 + |\text{Re} \{\tilde{\psi}_j\}|^2} \quad (8.8)$$

Ce critère permet de mesurer le "degré de complexité" du mode. Un mode réel a un index proche de 1. Un indice faible indique généralement un mode non-physique (dit de calcul) ou mal identifié. Plus rarement, on peut avoir un amortissement non-proportionnel significatif.

Un critère alternatif au MPC est d'étudier la phase moyenne des composantes du mode et la déviation standard associée. Son défaut est d'être aussi sensible aux composantes faibles (donc généralement mal identifiées) qu'au composantes fortes.

8.2.4 Réponse statique

Il est possible d'utiliser le résultat de l'identification pour estimer la réponse statique de la structure. En effet

$$[c] [\bar{K}]^{-1} [b] \approx [H(0)] = \sum_{j=1}^{NM} \frac{[c] \{\phi_{idj}\} \{\phi_{idj}\}^T [b]}{\omega_j^2} + [H_{id}^{res}(0)] \quad (8.9)$$

où le terme résiduel $[H_{id}^{res}(0)]$ doit être pris en compte afin d'inclure l'influence quasi-statique des modes haute fréquence.

Si le nombre de capteurs est suffisant pour pouvoir imposer $c = I$, une approximation de la flexibilité du modèle réduit sur les DDL de mesure est donnée par

$$[\bar{K}]^{-1} \approx \sum_{j=1}^{NM} \frac{\{\phi_{idj}\} \{\phi_{idj}\}^T}{\omega_j^2} \quad (8.10)$$

Cette approximation n'utilise pas de termes résiduels car l'hypothèse de réciprocité ne permet pas d'estimer les termes résiduels sur des fonctions de transfert non mesurées. Elle n'est donc valide que si un grand nombre de modes sont identifiés avec précision. D'un point de vue pratique, il semble donc plus sûr de se contenter de comparer les déformées statiques données par $H_{id}(0)$ que d'essayer de reconstruire une flexibilité voir une raideur statique expérimentale. L'intérêt de passer par l'identification est qu'en général $H_{id}(0) \neq H_{test}(0)$ car les capteurs ne permettent pas une mesure statique précise.

8.3 Beyond residuals : MIMO, reciprocity, real modes,

...

Hypotheses of multiplicity (existence of a unique pole in a MIMO test), reciprocity and existence of an underlying elastic model are often made. This section details methods used to build models verifying these hypotheses.

8.3.1 Multiplicity with assumed modal controllability

The most common approach (the method preferred by the author is detailed in the next section) consists in assuming poles and modal controllabilities $L_j = \psi_j^T b$ (also called mode participation factor) to be obtained by an algorithm capable of treating MIMO data sets (subspace, LSCE, ...). After selecting/optimizing the modes really useful in the model, one will as before use a linear least squares problem capable of accounting for residual terms. To obtain a global model having the same modal observability $c\psi_j$ for all inputs, one uses an a priori estimate of the modal participation factors to assemble transfer function blocs for each input

$$[\Phi(\lambda_j, \psi_j^T b)] \left\{ \begin{array}{c} [c\psi_1] \\ \vdots \\ [c\psi_{NW}] \\ [E] \end{array} \right\}_{(NM+NR \times NA) \times NS} = \left[\begin{array}{c} [H_{1,s}] \\ \vdots \\ [H_{NA,s}] \end{array} \right]_{(NA \times NW) \times NS} \quad (8.11)$$

with

$$[\Phi(\lambda_j, \psi_j^T b)] = \begin{bmatrix} \begin{bmatrix} \frac{\psi_1^T b_1}{i\omega_1 - \lambda_{[1:NM]}} \\ \vdots \\ \frac{\psi_1^T b_1}{i\omega_{NW} - \lambda_{[1:NM]}} \\ \vdots \\ \frac{\psi_1^T b_{NA}}{i\omega_{1:NW} - \lambda_{[1:NM]}} \end{bmatrix} & \begin{bmatrix} 1 & \frac{1}{-\omega_1^2} \\ \vdots & \vdots \\ 1 & \frac{1}{-\omega_{NW}^2} \\ \vdots & \vdots \\ 0 & \vdots \\ 0 & \vdots \end{bmatrix} & \begin{bmatrix} [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [E(s)] \end{bmatrix} \end{bmatrix}_{(NA \times NW) \times (NM + NR \times NA)} \quad (8.12)$$

From (8.11) one can clearly generate a linear list squares problem to find the modal observabilities and residual terms. Beyond the cost of assembling the matrices and solving a problem that can be large ($NA \times NW$ rows), this approach has at least two major problems. The poles and participation factors are obtained without residual terms but using computational modes. They are thus fundamentally biased. This bias is introduced in the least-squares problem and thus amplifies errors (to compensate for the bias). By construction, this method enforces an equal modeshape for each input. In practice, sequential tests, non-linearities, or simply measurement noise, lead to data that does not verify this property exactly. Avoiding the assumption during the least-squares problem thus allows to account for errors in resonances of individual FRFs without inducing bias in other modes.

8.3.2 Multiplicity with estimated residual

Pour un pôle unique, la matrice résidu doit pouvoir s'écrire comme le produit $\{c\psi_j\} \{\psi_j^T b\}$ qui est clairement de rang 1. Dans les rares cas où deux pôles sont suffisamment proches pour être confondus $\lambda_j \approx \lambda_{j+1}$, on peut rassembler les 2 résidus

$$\frac{R_j}{s - \lambda_j} = \frac{\{c\psi_j\} \{\psi_j^T b\} + \{c\psi_{j+1}\} \{\psi_{j+1}^T b\}}{s - \lambda_j} \quad (8.13)$$

et donc obtenir une matrice de rang 2.

L'approche proposée en Ref. [63, 77] commence par estimer les matrices de résidu pour l'ensemble des fonctions de transfert scalaires (paires entrée/sortie mesurée) en résolvant le problème (7.17). On estime donc une matrice résidu R_j qui pour un problème réellement MIMO a deux dimensions supérieures à 2 et qui à cause du biais d'estimation et du bruit de mesure est toujours de rang plein. On utilise alors la décomposition en valeurs singulières de $R_j U \Sigma V^T$ pour déterminer son rang effectif. Par définition de la SVD,

$$\left[\tilde{R}_j \right]_{NS \times NA} = \{U_1\}_{NS \times 1} \sigma_1 \{V_1\}_{NA \times 1}^T \quad (8.14)$$

est la meilleure approximation de rang 1 de R_j (au sens de la norme d'opérateur). De plus, le rapport σ_2/σ_1 donne une mesure de l'erreur relative faite en ignorant les autres valeurs singulières. En général le pôle est unique et, pour des tests MIMO, ce rapport donne une indication de la cohérence de l'identification (un rapport inférieur à 0.1 indique une très bonne identification). Dans de rares cas comme celui détaillé ci-dessous, un rapport élevé permet de détecter l'utilisation d'un seul pôle pour deux modes proches. Il est alors généralement possible de raffiner localement le modèle pour obtenir une estimation plus précise de ces deux pôles.

Un modèle pôle/résidu minimal est directement associé au modèle d'état à coefficient complexes

$$\begin{aligned} \left(s [I]_{2N \times 2N} - \begin{bmatrix} \lambda_j \\ \lambda_j \end{bmatrix} \right) \{\eta\} &= [\psi^T b] \{u\} \\ \{y\} &= [c\psi] \{\eta\} \end{aligned} \quad (8.15)$$

que l'on peut transformer en un modèle d'état à valeurs réelles (voir la fonction `res2ss` de la *SDT*).

8.3.3 Approximation du résidu : réciprocité

Le principe de réciprocité (dit de Maxwell-Betti) suppose l'égalité des fonctions de transfert d'une entrée en A vers la réponse en B et celle de l'entrée duale B vers la sortie duale en A (une réponse y et une force u sont duales si le produit uy est l'énergie apportée à la structure). En pratique, les tests MIMO ne sont jamais vraiment réciproques (car on ne fait qu'estimer le vrai transfert), mais il est souvent désirable de construire un modèle global réciproque (l'hypothèse de réciprocité permet de prédire des fonctions de transfert non-mesurées).

Au niveau des propriétés modales, l'hypothèse de réciprocité impose l'égalité des entrées et sorties modales duales (on dira collocalisées)

$$([c_{co1}] \{\psi_j\})^T = \{\psi_j\}^T [b_{co1}] \quad (8.16)$$

Pour des structures réciproque, la partie du résidu associé aux FRFs collocalisées doit donc être symétrique. Le degré de symétrie (différence entre R_{jco1} et $\hat{R}_{jco1} = (R_{jco1} + R_{jco1}^T)/2$) donne une indication de la qualité du modèle. L'hypothèse de minimalité peut aussi être imposée en calculant la décomposition en valeurs singulière de la partie symétrique qui vérifie $\hat{R}_{jco1} = U_{co1} \Sigma_{co1} U_{co1}^T$. Cette décomposition donne les entrées et sorties modales collocalisées

$$\{c_{co1} \psi_j\} = \{\psi_j^T b_{co1}\}^T = \sqrt{\sigma_{1co1}} \{U_{1co1}\} \quad (8.17)$$

En général, il y a beaucoup plus de capteurs que d'entrées, la décomposition (8.17) est donc utilisée pour déterminer les entrées modales (facteurs de participation). Dans le cas habituel où toutes les entrées ont un capteur collocalisé, les sorties sur les capteurs non collocalisés sont alors déterminées en utilisant un pseudo-inverse de $\{\psi_j^T b_{co1}\}$ au sens des moindres carrés $\{c\psi_j\} = [R_j] \{\psi_j^T b_{co1}\}^+$.

Pour la vérification pratique de la cohérence des résultats, on se rappellera que les résidus collocalisés de systèmes réciproques doivent vérifier la condition (8.5), car l'ajout de masse fait baisser les fréquences.

8.3.4 Real modes, elastic properties

For an elastic model with modal damping (viscous damping matrix Γ diagonal), the model in principal coordinates (3.42) can be rewritten as a rational fraction with truncation and residual terms

$$[\alpha(s)] = \sum_{j=1}^N \frac{\{c\phi_j\} \{b^T \phi_j\}^T}{s^2 + 2\zeta_j \omega_j s + \omega_j^2} + [E] + \frac{[F]}{s^2} = \sum_{j=1}^N \frac{[T_j]_{NS \times NA}}{s^2 + 2\zeta_j \omega_j s + \omega_j^2} + E(s) \quad (8.18)$$

This model representation where residues T_j are real and correspond to the contribution of normal modes has a linear dependency on residues and a non-linear dependency on poles. As for complex modes (7.17), one can thus determine $[T_j, E, F]$ by solving a linear least squares problem and consider an double minimization similar to (7.19) to solve the non linear optimization.

When assuming modal damping, the residue matrix T_j is directly associated to normal modes

$$[T_j] = \{c\phi_j\} \{\phi_j^T b\} \quad (8.19)$$

and multiplicity and reciprocity constraints can be dealt with as in the case of complex modes.

The implicit modeshape normalization in equations (8.18)-(8.19) corresponds to the mass normalization of normal modes used as default in these notes. The results remains valid for multiple modes where the residue matrix has a rank larger than 1.

If normal modes are not identified directly, the nominal result is as set of complex modes. The contribution of a given complex mode takes the form

$$\frac{[R_j]}{s - \lambda_j} + \frac{[\bar{R}_j]}{s - \bar{\lambda}_j} = \frac{2(s + \zeta_j \omega_j) \text{Re}(R_j) + 2\omega_j \sqrt{1 - \zeta_j^2} \text{Im}(R_j)}{s^2 + 2\zeta_j \omega_j s + \omega_j^2} \quad (8.20)$$

It thus appears that the complex mode residue $[R_j]$ associated with real modes $[T_j] = \{c\phi_j\} \{\phi_j^T b\}$ is purely imaginary and given by

$$[R_j] = \frac{[T_j]}{2i\omega_j \sqrt{1 - \zeta_j^2}} = [c] \frac{\{\phi_j\}}{\sqrt{2i\omega_j \sqrt{1 - \zeta_j^2}}} \frac{\{\phi_j\}^T}{\sqrt{2i\omega_j \sqrt{1 - \zeta_j^2}}} [b] \quad (8.21)$$

where it appears that the natural scaling for complex modes would result in real mode observabilities at a -45° angle.

One approach to estimate real elastic modes from complex modeshapes is thus to neglect the real part of R_j and use (8.21) to relate real and complex modeshape scaling. It is useful to recall that the generalized mass at the collocated sensor/actuator pair $c_{col} = b_{col}^T$ (see section 3.2.1) is then given by

$$\frac{1}{\mu_j(c_{col})} = c_{col} \phi_j \phi_j^T b_{col} = (c_{col} \phi_j)^2 = T_{jcol} = i\omega_j \sqrt{1 - \zeta_j^2} R_{jcol} \quad (8.22)$$

Typically an identification algorithm will capture the response level $R_j/2\zeta_j\omega_j^2$ at resonance correctly. A multiplicative error α on the damping level will thus induce the inverse error on the generalized mass.

A better approximation can be found using a non-proportional damping model. As shown in Ref [91], the complex modes of a minimal and reciprocal model are related to the mass, viscous damping and stiffness matrices by

$$M = (\tilde{\psi} \Lambda \tilde{\psi}^T)^{-1}, \quad C = -M \tilde{\psi} \Lambda^2 \tilde{\psi}^T M, \quad \text{and} \quad K = (\tilde{\psi} \Lambda^{-1} \tilde{\psi}^T)^{-1} \quad (8.23)$$

if and only if these complex modes are also proper. That is, they verify

$$\sum_{j=1}^{2N} \tilde{\psi}_j \tilde{\psi}_j^T = \tilde{\psi}_{N \times 2N} \tilde{\psi}_{N \times 2N}^T = [0]_{N \times N} \quad (8.24)$$

To obtain an estimate of the mass, damping and stiffness properties, it is thus desirable to proceed in two steps. One first finds a minimal and reciprocal approximation of the identified residues. One then determines c and $\tilde{\psi}$ such that $\tilde{\psi}$ verifies (8.24) and $c\tilde{\psi}$ is close to the $c\psi$ of the minimal reciprocal system. Using the complex modes $\tilde{\psi}$ and the identified poles λ , the model matrices can then be computed and transformed to principal coordinate form.

The validity of this transformation is linked to the fact that the considered group of modes must not be significantly coupled to other modes through damping. Such groups of modes can be detected using this frequency separation criterion typically used to validate the assumption of modal damping. Here however one seeks to validate that modes j within the group are separated from modes k outside the group by verifying

$$\frac{\max(\zeta_j \omega_j, \zeta_k \omega_k)}{|\omega_j - \omega_k|} \ll 1 \quad (8.25)$$

In practice, this identification of real modes requires very accurate measurements which are not very often found. The results in Ref. [91] are among the few documented such cases.

Chapitre 9

Test / analysis correlation

Direct uses of experimental models are relatively few. Notable exceptions are the synthesis of control systems and simple applications of *Structural Dynamics Modification* (SDM) (predictions of the response after a modification of the tested structure).

In general, one uses test results to validate a FEM model (correlation) and possibly correct its parameters (updating). Models can also be used to estimate the response at unmeasured DOFs (expansion) which can be used in conjunction with test derived models in SDM applications [92].

This chapter gives an broad overview of methods used in test / analysis correlation. Section 9.1 addresses topology correlation which is the initial fundamental step where the geometrical relation between the model and sensors is resolved to generate an observation equation. The section also addresses issues related to sensor placement.

Section 9.2 the introduces criteria defined for shapes known at sensors. One compares modal frequencies and shapes through the *Modal Assurance Criterion* MAC, the relative error on shape. Variants (COMAC, MACCO, ...) are also discussed.

Section 9.3 discusses criteria assuming knowledge of shapes on all DOFs of a mechanical model. In most cases, it is thus necessary to reduce the model to sensors (identified with DOFs in this case) or to estimate motion at all DOFs (one talks about expansion). Expansion and problems of spatial incompatibility are addressed in section 9.5.

Section focuses more specifically on criteria used for model updating. Finally section 9.6 discusses comparisons of transfer functions.

9.1 Topology correlation

9.1.1 Sensors, DOFs and observation matrices

One key concept of test/analysis correlation is the distinction between sensors and DOFs/states. To illustrate this concept, figure 9.1 shows a test configuration on a harp body. In this test, one measures translations in the line of sight of the vibrometer. The measurement $\{y\}$ performed define a set of *sensors* shown as arrows in the figure.

Experimental modal analysis (EMA) and parametric identification method lead to an input output model characterizing the response at sensors to loads applied on input locations. Typical sensors for EMA are

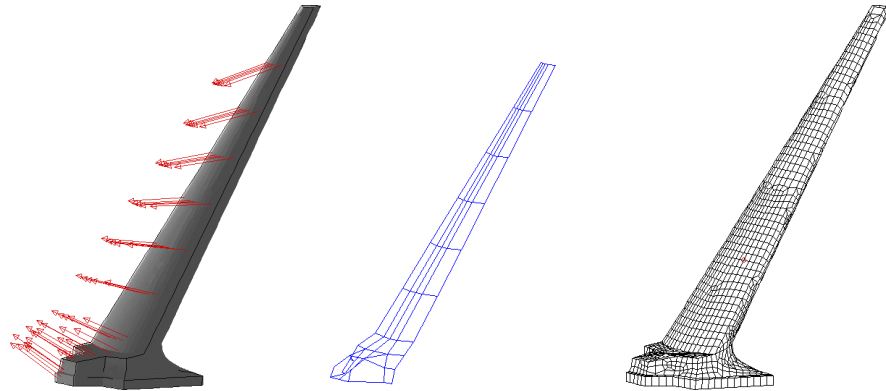


FIGURE 9.1 – Sensors, test wire-frame, FEM nodes

- accelerometers which measure at one location the acceleration in a given direction. Triaxial accelerometers are composed of three sensors in giving acceleration in orthogonal directions.
- laser vibrometers measure velocity or displacement in the line of sight.
- strain gages measure deformation in a given direction. They can be combined to estimate the surface strain tensor.

Analysis (typically performed using the Finite Element method), gives predictions at DOFs $\{q\}$ of the FEM model on the right of figure 9.1. FEM predictions and measurements are not directly comparable. Thus the first step of correlation, often called **topology correlation**, consists in building a function allowing to **predict FEM responses at sensors**. In most applications, DOFs and responses are linearly related so that one can write an observation equation of the form

$$\{y(t)\} = [c] \{q(t)\} \quad (9.1)$$

In the trivial case of sensors positioned at FEM nodes measuring in the direction of DOFs, the **observation matrix** $[c]$ is a boolean matrix often called localization matrix. It is however quite useful to consider other cases.

It is also useful to understand that the observation equation formalism applies in the same fashion for measurement of different nature (translation, rotation, deformation, temperature, ...) and that it is easily transposed to the case of inputs.

The generation of this observation matrix must deal with the following difficulties : sensors position generally do not coincide with FEM nodes and measurement directions are arbitrary. The two levels are dealt with separately. One first estimates 3D motion at test nodes, then projects onto the measurement directions.

Handling of measurement direction is simple. One just needs to project a 3D translation in the direction of the measurement. Some software packages use local coordinate systems to define the measurement direction. It is more general, and easier to handle, to give a measurement direction through 3 coordinates.

The optimal solution to account for test/FEM node non-coincidence is to use shape functions of the element formulation to predict the response at arbitrary locations. This is not always easy to implement in generic FEM codes that do not support this feature.

Early test/analysis correlation software considered the nearest node. This is often a poor solution and should be avoided whenever possible. Considering a rigid link between the nearest FEM node and the test node begins to be a reasonable alternative but causes problems for volume elements which do not have defined rotations or shells when drilling DOF is not implemented to provide consistent rigid body modes (drilling DOF is rotation around the normal to the element surface).

Difficulties with rotations can be avoided by estimating rotations and/or full motion based on the motion of multiple nodes of the element(s) the sensor is connected to. Such interpolations can be simple geometric weighting (RBE3 elements found in NASTRAN and other software) or be based on shape functions.

Figure 9.2 illustrates industrial topology correlation. In the engine cover example, green circles indicate links between FEM node positions and the nearest test node, additional links in blue indicate nodes used for rotation interpolation. In this example, the nearest node is not always on the proper surface so that detailed control of the matching algorithm is needed. The SNECMA example shows a test/FEM overlay for a vibrometer test. Accurate placement of the mesh within an image is often necessary for correlation of optical measurements.

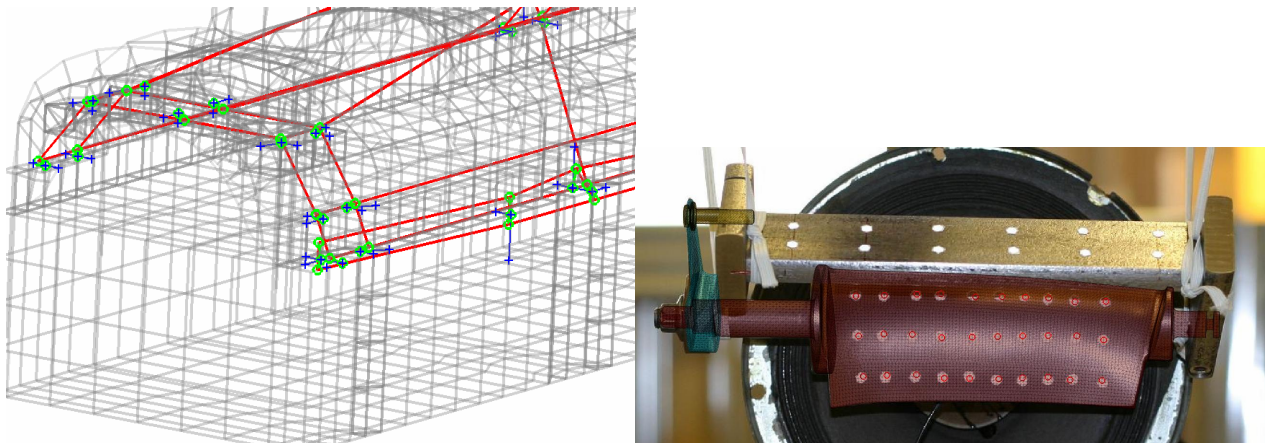


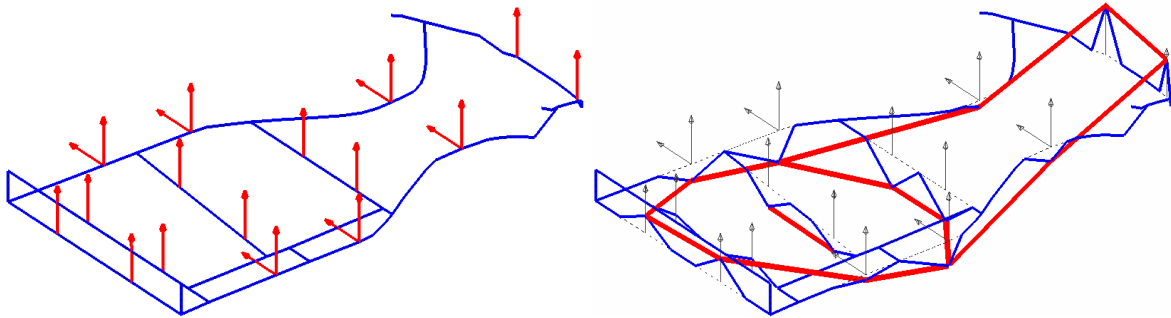
FIGURE 9.2 – Sample topology correlation. Left engine cover (Renault). Right stator blade (SNECMA)

9.1.2 Test design and topology correlation

When designing a test the wire-frame topology is designed with the following typical objectives :

- view deformations : needed to ensure test validity
- avoid distortions : needed for proper interpretation of test motion.
- prepare correlation : avoid points that are not modeled
- observe modes : geometrical independence of shapes. This criterion is the basis for most sensor placement algorithms.

The author's perspective on sensor placement is given in Ref. [36].



There are good and bad reasons for the non coincidence of test and FEM nodes.

- Sensors are placed with measurement objectives. They are thus placed by the test team on accessible positions, along regular lines to allow proper animation, ... (see wire-frame in figure 9.1). FEM nodes are typically placed by automatic meshers based on a CAD definition of the model (see figure 9.1). The coordination between test and analysis teams needed to enforce coincidence is not practical. For example the FEM mesh is generally finalized when the test is performed.
- Shell models are typically meshed on their mid-surface while the sensor is glued on the free surface. Proper correlation thus needs to account for the offset.
- The measurement of sensor locations may be imperfect. This is illustrated in figure 9.3 where one sees a point on the interior panel that are free in space. This is an experimental error and should be corrected using projections onto the FEM surface.

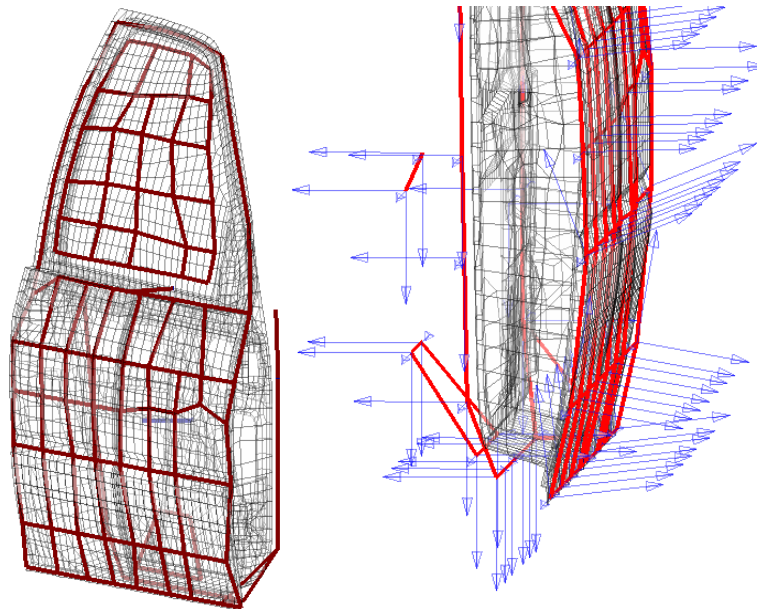


FIGURE 9.3 – Sample topology correlation problems : FEM and test meshes of a car door.

- The geometry of the test article may differ from the FEM geometry. This can be due to design changes that were not accounted for (in figure 9.3 the arm rest was simply not modeled) or, even in the best cases, to manufacturing tolerances. For assembled shell

structures typical of automotive applications, the best practice is to measure the actual test geometry within tolerances well below the shell thickness and to adjust the FEM geometry accordingly. Such adjustments have been shown to improve correlation greatly.

9.2 Correlating shapes known at sensors

9.2.1 Correlating modal frequencies

The simple correlation of linear system is the comparison of modal frequencies (poles). FEM models being generally elastic, one only compares frequencies using absolute

$$(\omega_{idj} - \omega_j) \quad (9.2)$$

or relative differences

$$\left(\frac{\omega_{idj} - \omega_j}{\omega_{idj}} \right) \quad (9.3)$$

Relative differences (expressed in %) are used more often since they are not dependent of the size of frequencies being compared.

A major difficulty as soon as errors become significant or modal density increases is the need to **pair modes**. Mode pairing consists in building a list of test/FEM modes that are sufficiently similar to be compared. Pairing requires a comparison of shapes, hence the introduction of shape criteria in the following sections.

9.2.2 Basic shape correlation (MAC)

The MAC (Modal Assurance Criterion [93, 94]) is the most widely used shape correlation criteria. Its implementation simplicity is probably a good reason for its success.

For two shapes U, V defined on the same sensors (same number of lines), the MAC is the **correlation coefficient** between the two vectors

$$\text{MAC}(U, V) = \frac{|\{U\}^H \{V\}|^2}{|\{U\}^H \{U\}| |\{V\}^H \{V\}|} \quad (9.4)$$

In the case of modes (initial application for the MAC), one compares the experimental deformation ($c\phi_{idj}$) (modal observability on sensors) and the observation of the FEM mode shapes on the same sensors $[c]_{NS \times N} \{\phi_j\}_{N \times 1}$ (see section 9.1.1 for the construction of c). MAC expression thus becomes

$$\text{MAC}_{jk} = \frac{\left(\{c\phi_{idj}\}^T [c] \{\phi_k\} \right)^2}{\left(\{c\phi_{idj}\}^T \{c\phi_{idj}\} \right) \left(\{\phi_k\}^T [c]^T [c] \{\phi_k\} \right)} \quad (9.5)$$

A perfect correlation between two modes, gives a MAC equal to 100% (or 1). A MAC equal to 1 implies the equality of the subspaces generated by two vectors. Typical choices are

$$\begin{array}{ll}
100\% \leq \text{MAC} \leq 90\% & \text{Correlated modes} \\
90\% \leq \text{MAC} \leq 70\% & \text{Doubtfull correlation} \\
10\% \leq \text{MAC} \leq 70\% & \text{Uncorrelated modes} \\
\text{MAC} \leq 10\% & \text{Modes are nearly orthogonal}
\end{array} \tag{9.6}$$

The MAC does not depend on vector scaling, thus test and FEM nodes that are normalized using two different strategies can be compared directly. In other words, the MAC is independent of the modal mass.

In some applications, in particular when using sensors of different nature (translation, rotation, strain, ...), it is useful to weigh the influence of each sensor (use a non Euclidian norm). One thus uses a MAC of the form

$$\text{MAC}_{W,ij} = \frac{\left(\{\phi_{i\text{di}}\}^T [W] \{\phi_j\} \right)^2}{\left(\{\phi_{i\text{di}}\}^T [W] \{\phi_{i\text{di}}\} \right) \left(\{\phi_j\}^T [W] \{\phi_j\} \right)} \tag{9.7}$$

where the weighting matrix $[W]$ is typically symmetric, positive definite. A classical choice is to use the mass $[W] = [M]$ (possibly reduced at sensors), the resulting norm being the kinetic energy norm. There is then a very obvious link between the MAC and the orthogonality criteria treated later.

In general, one computes the MAC for all associated pairs and shows results as a table (as in table 9.1) or figure (see 9.4).

TABLE 9.1 – Auto-MAC (in %) of GARTEUR testbed [95] for test with 24 sensors

#	1	2	3	4	5	6	7	8
1	100.	0.0	0.2	1.6	0.5	18.1	0.0	0.1
2	0.0	100.	0.5	0.2	0.0	0.1	0.1	0.0
3	0.2	0.5	100.	0.3	11.8	1.7	0.5	0.1
4	1.6	0.2	0.3	100.	1.1	1.8	0.1	2.0
5	0.5	0.0	11.8	1.1	100.	0.8	21.3	0.0
6	18.1	0.1	1.7	1.8	0.8	100.	0.5	0.4
7	0.0	0.1	0.5	0.1	21.3	0.5	100.	0.4
8	0.1	0.0	0.1	2.0	0.0	0.4	0.4	100.

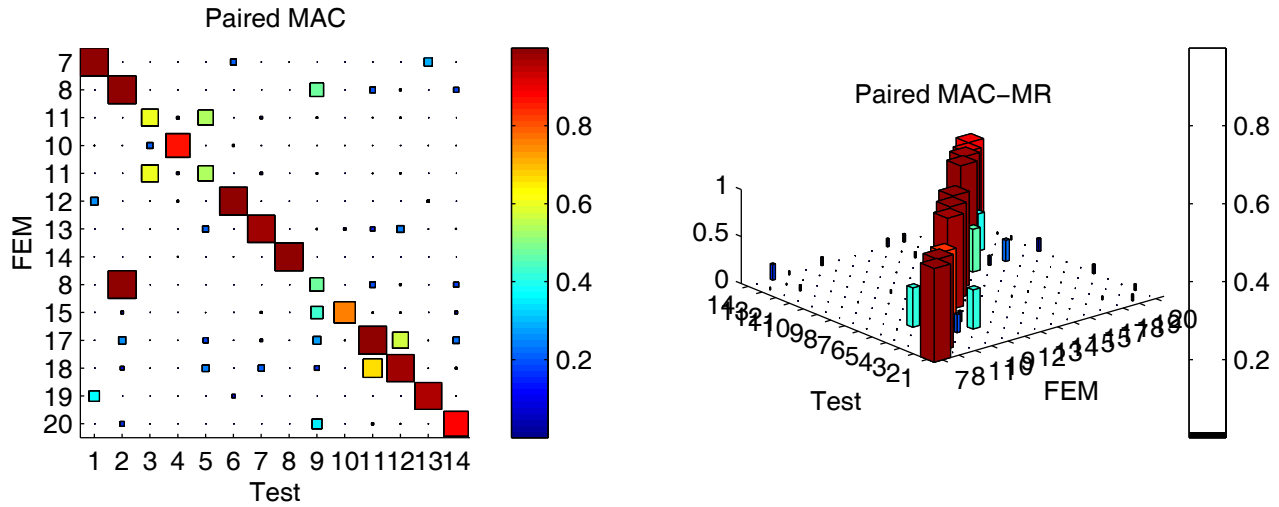


FIGURE 9.4 – MAC and MAC-MR after mode pairing

A priori, only the terms associated with paired shapes have a meaning. In practice however, when the shapes being compared are modes and the number of sensors is sufficient, shapes are relatively orthogonal. In reality, modes are only mass orthogonal. So using a reduced mass may be needed, but may not be sufficient. It is then necessary to improve sensor placement [36].

In practice, it is desirable, before testing, to verify on the initial FEM that off-diagonal terms are below 10% (on the auto-MAC). In table 9.1, one thus sees that the MAC comparison of modes 1-6, 3-5, 5-7 have non negligible values. By looking the shapes, one sees similar deformations. Two node bending (mode 1) and 4 node bending mostly differ by fuselage motion, ...

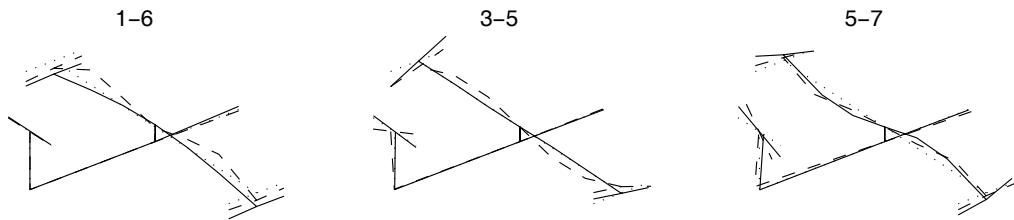


FIGURE 9.5 – Paires de déformées superposées pour le maillage expérimental de la maquette GARTEUR SM-AG-19

9.2.3 Erreurs sur les déformées

Lorsque l'on dispose de modes normés de la même façon, il est beaucoup plus précis de comparer les vecteurs composante par composante. Par exemple, on définira l'erreur relative sur les observabilité modales par

$$e_j = \frac{\| \{c\phi_{jid}\} - [c] \{\phi_{jB}\} \|}{\| \{c\phi_{jid}\} \| + \| [c] \{\phi_{jB}\} \|} \tag{9.8}$$

où la norme utilisée $\| \cdot \|$ reste à préciser. Les normes les plus souvent utilisées sont les normes de 1, 2 et ∞ :

$$\begin{aligned} \|\phi_j\|_1 &= \sum_{i=1\dots N} |\phi_{ij}| \\ \|\phi_j\|_2 &= \sqrt{\sum_{i=1\dots N} |\phi_{ij}|^2} \\ \|\phi_j\|_\infty &= \max_{i=1\dots N} |\phi_{ij}| \end{aligned} \quad (9.9)$$

où le choix n'est souvent pas neutre pour les conclusions tirées de la comparaison. On notera de plus que ces normes ont un sens pour comparer le comportement entrée/sortie (deux modèles ayant des mêmes pôles, observabilités et commandabilités modales très proches ont des comportement dynamiques très proches) mais ne permettent pas de dire grand chose sur le comportement mécanique.

La mesure de corrélation (9.8) est beaucoup plus stricte que le critère MAC et ne vaut zéro que si les traces des modes sur les degrés de liberté considérés sont exactement égales. On utilise ici une norme et non plus une mesure de corrélation. **En pratique, une erreur relative en forme propre inférieure à 10% dénote une très bonne correspondance** (en général, on dit *corrélation* ce qui n'est pas très bien choisi).

L'erreur relative en composantes suppose que les modes soient normalisés de la même façon. Si ce n'est pas le cas, il est possible d'utiliser le facteur d'échelle modale (MSF pour **modal scale factor**)

$$\text{MSF}_j = \frac{\{c\phi_{\text{id}j}\}^T [c] \{\phi_j\}}{[c] \{\phi_j\}^T [c] \{\phi_j\}} \quad (9.10)$$

afin de représenter les deux modes sur la même échelle. Dans le cas choisi ci-dessus, le mode analytique $\{\phi_j\}$ serait multiplié par le facteur d'échelle (9.10) avant d'être utilisé dans le critère (9.8). Le facteur d'échelle peut aussi être utilisé pour normaliser les modes avant utilisation du COMAC.

Comme toujours, il peut être utile de considérer ce critère après expansion sur tout les DDLs. Pour un mode étendu, l'erreur relative sur les déformées modales devient

$$e_j = \frac{\|\{\phi_{\text{exp}}(c\phi_{\text{id}j})\} - \{\phi_j\}\|}{\|\{\phi_{\text{exp}}(c\phi_{\text{id}j})\}\| + \|\{\phi_j\}\|} \quad (9.11)$$

Comme pour le MAC, il est alors approprié de choisir une norme à pondération physique : la semi-norme en énergie de déformation $\|v\|_K^2 = \{v\}^T [K] \{v\}$ ou la norme en énergie cinétique $\|v\|_M^2 = \{v\}^T [M] \{v\}$. Il faut noter cependant qu'il y a alors couplage entre erreur d'identification et erreur d'expansion liée à l'inexactitude du modèle.

9.2.4 Localization of poor correlation

This section should be rewritten on the basis of [75, 96] which gives more up to date examples.

The MACCO criterion consists in the sequential elimination of sensors that contribute most to the poor correlation. It is implemented by computing the MAC while removing one sensor at

a time. The sensors whose removal improves correlation best is then removed and the procedure continued [75, 96]. The sorting can be used to analyze modeling or test errors.

The earlier COMAC (**CO**ordinate **MO**dal **AS**surance **C**riterion [97]) sought to identify problematic sensors but we did not find it very convincing over the years.

One last common strategy is to analyze the error in terms of sensor subsets. For c_Ω the observation matrix of a sensor set Ω one can define the MAC by subset as

$$\text{MAC}(\Omega)_{ij} = \frac{\left(\{c_\Omega \phi_{\text{id}j}\}^T [c_\Omega] \{\phi_j\} \right)^2}{\left(\{c_\Omega \phi_{\text{id}j}\}^T \{c_\Omega \phi_{\text{id}j}\} \right) \left(\{\phi_j\}^T [c_\Omega]^T [c_\Omega] \{\phi_j\} \right)} \quad (9.12)$$

This can be used to identify substructures that have poor correlation. This is for example useful when the structure is composed of parts with notably different amplitudes/stiffness. It is then useful to weight sensors differently and considering subsets is an approximation of that. Considering a reduced model norm is another.

9.3 Model based correlation

Quand on dispose d'un modèle mécanique, il est souhaitable de l'utiliser définir des critères de comparaison ayant une motivation mécanique plutôt que de conserver les critères purement entrée/sortie définis à la section précédente.

Comme détaillé dans les sections suivantes, les principaux critères à motivation mécanique sont basés sur la vérification des équations d'équilibre et l'orthogonalité des vecteurs propres par rapport à la masse ou la raideur.

L'application de ces critères suppose la connaissance des résultats expérimentaux sur l'ensemble des DDL du modèle. Ce problème de compatibilité spatiale est résolu par réduction du modèle ou par estimation des réponses sur l'ensemble des DDL abordé en section 9.5.

9.3.1 Critères sur les équations d'équilibre

Les quantités comparées sont définies du côté analytique par des équations d'équilibre. Ainsi, les modes normaux du modèle élément fini sont solutions du problème aux valeurs propres

$$([K] - \omega_j^2 [M]) \{\phi_j\} = \{0\} \quad (9.13)$$

où le zéro du membre de droite traduit l'équilibre du système en l'absence de sollicitation extérieure. Cependant, cet équilibre est violé lorsque les fréquences et formes propres identifiées sont substituées aux quantités analytiques dans l'équation (9.13). Il vient alors un **résidu modal** (où effort résiduel modal)

$$\{R_j\} = ([K] - \omega_{\text{id}j}^2 [M]) \{\phi_{\text{id}j}\} \quad (9.14)$$

qui est d'autant plus faible que l'équation d'équilibre est bien vérifiée. Pour des pour des réponses en fréquence (ou statique en $\omega = 0$), on définit de manière similaire le résidu

$$\{R_j\} = Z(\omega) \{q_{\text{id}}(\omega)\} - [b] \{u_{\text{id}}(\omega)\} \quad (9.15)$$

Les résidus d'erreur peuvent être exploités de plusieurs manières. Leur norme donne une indication scalaire de la corrélation. La distribution de l'amplitude sur la structure donne une idée de la localisation des erreurs.

Comme toujours, le choix d'une bonne norme est essentiel. Sur un espace fini, le choix d'une norme correspond à la sélection d'une matrice de pondération W conduisant à

$$\|R_j\|_W = \{R_j\}^T [W] \{R_j\} \quad (9.16)$$

Les résidus correspondent généralement à des efforts. Pour exprimer l'erreur en termes d'énergie, ce qui est toujours le bon choix en mécanique, il convient donc d'associer un déplacement au résidu. L'approche la plus simple est de considérer la réponse statique au résidu $W = \hat{K}^{-1}$, on étudie alors l'énergie de déformation associée au résidu. Cette norme correspond à la norme en énergie de déformation ($W = K$) calculée sur le résidu en déplacement

$$\{\bar{R}_j\} = [\hat{K}^{-1}] \{R_j\} \quad (9.17)$$

L'exploitation des résidus en déplacement sera faite en sections 11.2.3.

Il existe de nombreuses manières de définir et étudier des résidus. Les différences entre chaque résultat sont souvent instructives et il est donc utile de comparer plusieurs méthodes. Brièvement, voici quelques variantes importantes.

Dans le cas d'une structure avec modes de corps rigide, K est singulier. $[\hat{K}^{-1}]$ doit donc être un pseudo-inverse de K . Les pseudo-inverses standards sont la réponse statique des modes flexibles et le shift en masse (voir [16, 44] pour plus de détails). On notera que la pertinence du critère d'erreur n'exige pas que \hat{K} soit la vraie raideur. Cette constatation est importante en recalage (où l'on cherche justement à corriger le modèle) mais permet aussi de considérer des approximations de K conduisant à des temps de calculs plus faibles tout en utilisant une norme beaucoup plus judicieuse que la norme euclidienne $W = I$.

Le résidu en déplacement peut aussi être défini de la manière suivante dérivée des travaux sur l'erreur en relation de comportement (travaux au LMT de l'ENS Cachan initialement introduits pour étudier la convergence de modèles élément fini et aussi appliqués au recalage [98]). On peut trouver $\{\psi_i\}$ tel que l'équilibre soit satisfait si l'on "relâche" le fait que ce vecteur doit être un mode propre

$$[K] \{\hat{\phi}_j\} - \omega_j^2 [M] \{\phi_{jia}\} = \{0\} \quad (9.18)$$

Le résidu caractérisant l'erreur de modélisation est alors la différence $\{\phi_j\} - \{\hat{\phi}_j\}$ qui, ici, est clairement égal au résidu en déplacement (9.17). Cette nouvelle formulation du résidu en déplacement est cependant intéressante par ses nombreuses variantes [98]. On peut permuter ϕ et $\hat{\phi}$ ce qui conduit à une inversion de la masse plutôt que de la raideur (mais la masse est souvent non-inversible dans les codes élément fini). On peut introduire plus facilement, un résidu associé à l'erreur sur le modèle d'amortissement, ...

9.3.2 Critères basés sur les relations d'orthogonalité

Les relations d'orthogonalité (3.36)-(3.37) peuvent être utilisées afin de vérifier la corrélation calcul-essai. On rappelle que les modes propres vérifient les relation

$$\{\phi_j\}^T [M] \{\phi_k\} = \mu_j \delta_{jk} \quad \{\phi_j\}^T [K] \{\phi_k\} = \mu_j \omega_j^2 \delta_{jk} \quad (9.19)$$

où la normalisation en masse modale unitaire (soit $\mu_i = 1$) est utilisée pour l'ensemble de ce cours.

Si l'on suppose que l'ensemble des degrés de liberté sont mesurés ou encore que les matrices éléments finis sont condensées sur le sous-ensemble des capteurs, il est possible de substituer dans les équations (9.19) les quantités identifiées aux paramètres modaux analytiques. On obtient une comparaison entre vecteurs propres analytiques et identifiés, appelée POC (*Pseudo Orthogonality Check*) or CGM (*Cross-Generalized Mass*) pour la masse et CGK (*Cross-Generalized Stiffness*) pour la raideur, et donnée par

$$\mu_{jk} = \{\phi_{idj}\}^T [M] \{\phi_k\} \quad \kappa_{jk} = \{\phi_{idj}\}^T [K] \{\phi_k\} \quad (9.20)$$

où, pour des modes normés en masse, on s'attend à avoir $\mu_{jk} \approx \delta_{jk}$ et $\kappa_{jk} \approx \omega_j^2 \delta_{jk}$. On notera que pour des vecteurs non normés, on pourra diviser par la norme des vecteurs ce qui donne la racine carrée du MAC pondéré en masse.

Pour la corrélation et la localisation d'erreur, ces critères sont moins populaires que ceux sur les vecteurs connus sur les capteurs (surtout le MAC et ses variantes) car il faut d'une part résoudre le problème de compatibilité des tailles entre modèles analytique et expérimental et d'autre part savoir interpréter les termes non nuls qui apparaissent souvent lorsque l'on compare deux modes différents ($i \neq j$).

L'orthogonalité par rapport à la masse est cependant utilisée dans certaines industries (le spatial américain en particulier) comme indicateur contractuel de la validation d'un modèle analytique.

Pour le recalage, certains ont aussi considéré un test sur l'orthogonalité des formes propres identifiées par rapport aux matrices éléments finis comme

$$\{\phi_{idj}\}^T [M] \{\phi_{idk}\} \approx \mu_{idj} \delta_{jk} \quad \{\phi_{idj}\}^T [K] \{\phi_{idk}\} \approx \mu_{idj} \omega_{idj}^2 \delta_{jk} \delta_{jk} \quad (9.21)$$

La méthode de recalage associée est dite *méthode des énergies modales* (vous vous convaincrez que $\{\phi_{idj}\}^T [M] \{\phi_{idk}\}$ correspond à une énergie cinétique, et $\{\phi_{idj}\}^T [K] \{\phi_{idk}\}$ à une énergie de déformation).

9.4 Objective functions for updating

Comparing frequencies

The simplest objective function for updating is the quadratic norm on frequency errors. The computation of frequency sensitivities being well mastered, updating based on frequencies is easily implemented. The application of a sensitivity method leads to iterations associated with the least squares problem

$$\min_{\delta p} \left\| \left[\frac{\partial \omega_j^2}{\partial p_1} \cdots \frac{\partial \omega_j^2}{\partial p_{NP}} \right] \begin{Bmatrix} \delta p_1^{n+1} \\ \vdots \\ \delta p_{NP}^{n+1} \end{Bmatrix} - \left\{ \omega_{\text{id}_j}^2 - \omega_j^2(p^n) \right\} \right\| \quad (9.22)$$

In practice,

- modes need to be paired to allow the frequency error. The trivial approach that takes the nearest frequency is often poor. The use of a pairing based on maxima of the MAC is often sufficient. In presence of high modal densities or of modes varying over a large range of frequencies, as shown in 9.6, pairing may become a critical issue.
- problème (9.22) est généralement mal ou très mal conditionné dès que le nombre de variables à optimiser est significatif (par rapport au nombre de modes identifiés). Les outils numériques pour améliorer le conditionnement ont été évoqués en section 11.5 mais il convient le plus souvent de chercher à augmenter le nombre de données expérimentales (en prenant en compte les déformées modales, ...) et de diminuer le nombre de paramètres (en sélectionnant les paramètres les plus influents, ...)
- il est souhaitable de donner une influence similaire aux différentes fréquences et donc d'utiliser l'erreur relative sur les fréquences
- une extension parfois mentionnée est l'idée d'utiliser aussi les fréquences des anti-résonnances

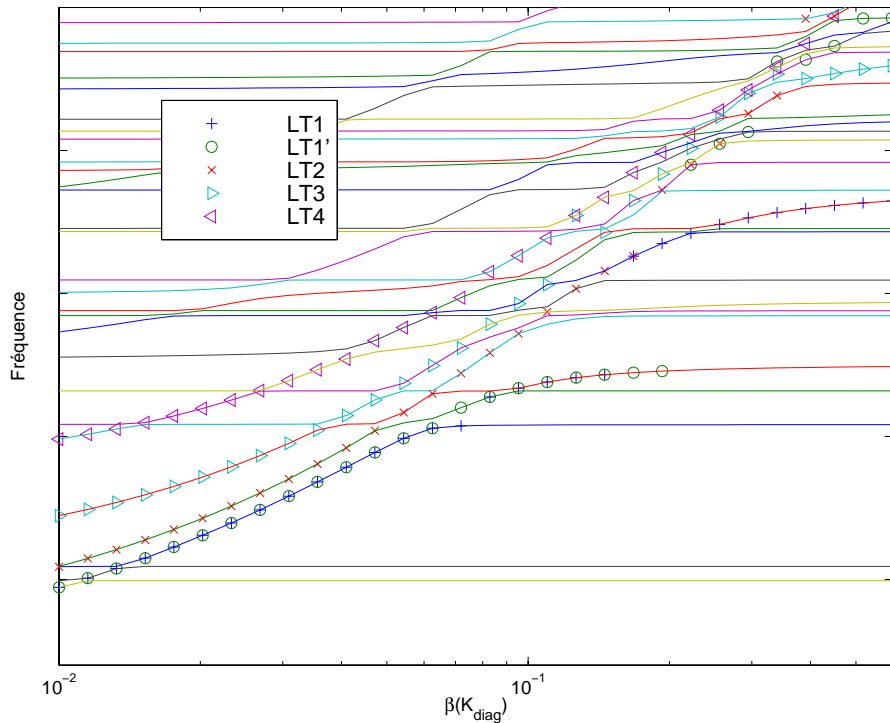


FIGURE 9.6 – Évolution des fréquences appariées en fonction d'un paramètre physique. (Modèle MD-EPC Ariane 5).

Fréquence et déformées connues sur les capteurs

Pour prendre en compte les déformées, on pourra utiliser une combinaison erreur en fréquence + erreur en déformée (erreur relative (9.8) avec ou sans facteur d'échelle, un moins la diagonale du MAC (9.4), MAC pondéré en masse, etc.). On pourra par exemple définir un résidu dont la norme est à minimiser par

$$R(p) = \sum_j \alpha_j \left\{ \begin{array}{c} \left(\frac{f_{FE,i}^2 - f_{T_{est},j}^2}{f_{T_{est},j}^2} \right) \\ \beta_j (1 - MAC_{i,j,M,capt}) \end{array} \right\} \quad (9.23)$$

La formulation étant donnée en terme de résidu, on se reportera à 11.5.1 pour la discussion sur les algorithmes d'optimisation. Le calcul de la sensibilité du MAC est pénible à formuler

$$\begin{aligned} \frac{\partial MAC_{i,j}}{\partial p} = & 2 \frac{\left[\left\{ c \frac{\partial \phi_{FE,i}}{\partial p} \right\}^T [M_{r0}] \{ \phi_{T_{est},j} \} \right] \left[\{ c \phi_{FE,i} \}^T [M_{r0}] \{ \phi_{T_{est},j} \} \right]}{\| \{ c \phi_{FE,i} \} \|_{M_{r0}}^4 \| \{ c \phi_{T_{est},j} \} \|_{M_{r0}}^4} \\ & - 2 \frac{\| \phi_{T_{est},j} \|_{M_{r0}}^2 \left[\frac{\partial c \phi_{FE,i}}{\partial p} \right]^T M_{r0} c \phi_{FE,i} \left[c \phi_{FE,i}^T M_{r0} c \phi_{FE,i} \right] \left[c \phi_{FE,i}^T M_{r0} \phi_{T_{est},j} \right]^2}{\| \{ c \phi_{FE,i} \} \|_{M_{r0}}^4 \| \{ c \phi_{T_{est},j} \} \|_{M_{r0}}^4} \end{aligned} \quad (9.24)$$

mais rapide à calculer une fois programmé.

La formulation est sensible au poids accordé à chaque mode α_j et au poids relatif β_j entre les critères sur les fréquences et les MAC. En pratique, on pourra prendre $\alpha_j = 1$ et β_j de manière à donner le même poids aux deux parties du résidu, mais il est difficile de justifier ce choix.

Les difficultés d'appariement et de conditionnement/régularisation sont toujours importants.

Orthogonalité, énergies modales

Les relations d'orthogonalité fournissent des critères simples de mise en oeuvre et de faible dimension. Parmi les nombreuses méthodes possibles, on notera en particulier la méthode des **énergies modales** (développée par le LMARC de Besançon). Les conditions d'orthogonalité vérifiées par les modes normaux conduisent à la relation

$$E_{Cjj} \omega_j^2 - E_{Pjj} = 0 \quad (9.25)$$

Supposant qu'une réduction du modèle ou une expansion des modes expérimentaux (voir section 9.5) permet de définir les résultats de test sur l'ensemble des DDLs éléments finis, on définit les énergies modales expérimentales par

$$E_{Cjj} = \{ \phi_{idj} \}^T [M(p)] \{ \phi_{idj} \} \quad \text{et} \quad E_{Pjj} = \{ \phi_{idj} \}^T [K(p)] \{ \phi_{idj} \} \quad (9.26)$$

Les paramètres du modèle sont alors corrigés de manière à minimiser l'erreur sur la relation (9.25). Pour une méthode de sensibilité, on considère donc des itérations de la forme

$$\left[\frac{\partial (E_{Cjj} \omega_{idj}^2 - E_{Pjj})}{\partial p} \right] \{ \delta p^{n+1} \} + \{ E_{Cjj} \omega_{idj}^2 - E_{Pjj} \} = \{ 0 \} \quad (9.27)$$

Pour limiter le coût numérique de cette approche, on peut négliger les termes liés à la dépendance des déformées modales étendues aux paramètres, ce qui conduit à la formulation simplifiée

$$\{\phi_{id_j}\}^T \left[\frac{\partial Z(\omega_{id_j})}{\partial p} \right] \{\phi_{id_j}\} \{\delta p^{n+1}\} + \{E_{Cjj}\omega_{id_j}^2 - E_{Pjj}\} = \{0\} \quad (9.28)$$

et à des itérations définies par

$$\{\phi_{id_j}\}^T \left(\left[\frac{\partial(Z(\omega_{id_j}))}{\partial p} \right] \{\phi_{id_j}\} \{\delta p^{n+1}\} + [Z(\omega_{id_j})] \{\phi_{id_j}\} \right) = \{0\} \quad (9.29)$$

qui fait clairement apparaître la multiplication à gauche par $\{\phi_{id_j}\}^T$ comme la recherche d'une solution particulière du problème de moindres carrés associé au résidu modal $\{R_j(p)\} = ([K(p)] - \omega_{id_j}^2 [M(p)]) \{\phi_{ex_j}\}$.

Contrairement aux méthodes travaillant directement sur les fréquences et déformées, la méthode des énergies modales ne nécessite pas l'appariement.

Résidus dynamiques et erreurs en énergie

Les résidus en énergie se prêtent aussi au processus de recalage. Pour aider la programmation, il est utile de reformuler (9.37). La fonction objectif à minimiser est donnée comme solution de

$$J_K(p) = \sum_{i=1}^{NM} J_j(p) = \sum_{i=1}^{NM} \min_{\phi_j, exp} \|R'_j(p)\|_{Q_{0,j}} \quad (9.30)$$

où $R'_j(p) = \begin{Bmatrix} Z(\omega_{Test,j}, p)\phi_{exp}(\omega_{Test,j}, p) \\ c\phi_{exp}(\omega_{Test,j}, p) - y_{Test}(\omega_{Test,j}) \end{Bmatrix}$ et $[Q_{0,j}] = \begin{bmatrix} \hat{K}(p_0) & 0 \\ 0 & \gamma_j [Q_j] \end{bmatrix}$ avec $[Q_j]$ la matrice de pondération sur les capteurs introduite pour tenir compte de la confiance accordée à chaque mesure.

Il est possible de reformuler le problème. En effet, plutôt que d'utiliser l'opérateur \hat{K} , on peut définir le résidu en déplacement par la relation $[K] \{R_{D,j}\} - [Z(\omega_{Test,j})] \{\phi_{exp,j}\} = 0$ et introduire un multiplicateur de Lagrange λ_j pour ajouter cette contrainte à la formulation du problème de minimisation (9.37). On obtient

alors

$$\begin{aligned} J(\omega_{Test,j}) = & \min_{R_{D,j}, \phi_{exp,j}, \lambda_j} \left(\{R_{D,j}\}^T K \{R_{D,j}\} \right. \\ & + \gamma_j \|c \{\phi_{exp,j}\} - \{y_{Test,j}\}\|_{Q_j}^2 \\ & \left. + \lambda_j (K \cdot \{R_{D,j}\} - Z(\omega_{Test,j}) \{\phi_{exp,j}\}) \right). \end{aligned} \quad (9.31)$$

A l'optimum, la dérivée de $J(\omega_{Test,j})$ par rapport à $R_{D,j}$, $\phi_{exp,j}$ et λ_j doit être nulle, ce qui implique

$$\begin{bmatrix} K & 0 & K \\ 0 & \gamma_j c^T Q_j c & -Z_j \\ K & -Z_j & 0 \end{bmatrix} \begin{Bmatrix} R_{D,j} \\ \phi_{exp,j} \\ \lambda_j \end{Bmatrix} = \begin{Bmatrix} 0 \\ \gamma_j c^T Q_j y_{Test,j} \\ 0 \end{Bmatrix}, \quad (9.32)$$

où $Z_j = Z(\omega_{Test,j})$ est la raideur dynamique calculée aux fréquences expérimentales.

La première ligne de cette équation donnent $\lambda_j = -R_{D,j}$, de sorte que les inconnues se résument aux deux champs $R_{D,j}$ et $\phi_{exp,j}$ (voir [99]). Le problème que l'on cherche à résoudre est donc

$$\begin{bmatrix} -K & Z_j \\ Z_j & \gamma_j c^T Q_j c \end{bmatrix} \begin{Bmatrix} R_{D,j} \\ \phi_{exp,j} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \gamma_j c^T Q_j y_{Test,j} \end{Bmatrix}. \quad (9.33)$$

Il est important de noter que la résolution de (9.33), en dehors du fait qu'elle ne nécessite pas l'inversion de l'opérateur \hat{K} , est plus rapide que la résolution du problème équivalent à un champ (9.37). Le fait que le système (9.33) soit symétrique permet en outre l'emploi de méthodes de résolutions particulières. Pourtant cette résolution est très coûteuse et donc accessible que par des méthode de réduction [100].

Comparaison avec les FRFs

L'utilisation de réponses en fréquences (fonctions de transfert) peut être avantageuse dans des situations où il est important de prendre en compte l'amortissement, surtout si celui-ci est non proportionnel (ce qui demande de distinguer modes normaux et complexes). Parmi les avantages et inconvénients, on peut citer

- + Recalage de modèles amortis ;
- + Capacité à traiter les fortes densités modales ;
- + Recalage dans le domaine "moyenne" fréquence ;
 - temps de calcul des fonctions de transfert et de leur sensibilités à partir du modèle ;
 - difficulté de sélectionner les fréquences pour la comparaison
 - détection et analyse des problèmes de mesure

La comparaison peut se faire au niveau des réponse mesurées (approches directes liés et fonctions objectifs (9.40)-(9.41)) ou du résidu dynamique (9.39) couplé à un problème d'expansion (9.37).

9.5 Expansion : estimation based on a model

9.5.1 Motivation and classification

Given an observation equation $\{y\} = [c] \{q\}$, one can compare test and analysis results (modes, FRF, etc.) on the given sensors.

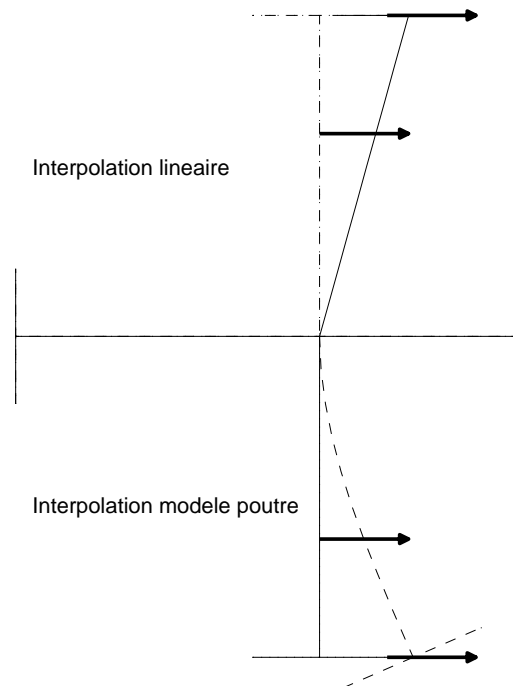
The first use of this comparison is to allow the superposition of analytical and test results on the wire-frame representation of the 3D structure (as shown in figure 9.1).

The direct viewing of the measured response on an experimental pseudo-mesh is essential for test validation. However for test that do not use triaxial sensors, the motion of experimental nodes is not fully measured. For animation, one thus starts by assuming that the motion in non-measured direction is zero.

Souvent, ceci ne permet pas de bien connaître/comprendre le mouvement car la densité spatiale de mesure est trop faible. On cherche donc à interpoler le mouvement 3-D des noeuds expérimentaux, dans un premier temps, puis le mouvement de tous les DDLs du modèle élément fini.

Historiquement, les premières méthodes d'expansion étaient basées sur des **interpolations géométriques** simples (bases linéaires, quadratiques, etc.). On s'est rapidement aperçu que les bases de fonctions de forme élément fini permettaient de créer rapidement des bases d'interpolations efficaces puis par extension que des déformées particulière d'un modèle élément fini initial donnaient des résultats plus précis tout en permettant le traitement de géométries très complexes. Pour les méthodes d'interpolation basées sur un modèle élément fini, on parle généralement d'**expansion**.

La figure ci-dessous illustre pour la maquette GARTEUR qu'une interpolation linéaire de la flexion dans le plan à partir de 3 capteurs donne une idée correcte mais moins précise qu'une interpolation cubique (celle utilisée par les éléments de poutre d'Euler-Bernoulli). En particulier, l'interpolation basée sur le modèle poutre fait apparaître un couplage en entre les translations horizontales mesurées et les translations verticales liée à la rotation du bout de l'aile.



On peut classifier les méthodes d'expansion en les considérant comme des méthodes d'estimation. Les méthodes d'estimation combinent des résultats de mesure (ici un déplacement connu sur des capteurs $\{y_{test}\}$) et des informations initiales (ici un modèle élément fini).

L'erreur sur les données d'essais est donnée par

$$\epsilon = \|\{y_{Test}\} - [c] \{q_{ex}\}\|_Q^2 \quad (9.34)$$

où le choix de la norme Q est un problème important. Le remplacement de l'identité (norme Euclidienne) par norme prenant en compte l'incertitude sur les composantes de y_{Test} semble une solution particulièrement adaptée. Des mesures en énergie ont été considérées en [98], mais leur motivation n'est pas claire.

Deux approches essentielles ont été considérées pour prendre en compte le modèle. Les méthodes de **sous-espaces** utilisent le modèle pour construire un base de réduction comprenant

au plus autant de vecteurs que de capteurs. Les méthodes basées sur un **modèle d'erreur** utilisent un critère de validité du résultat étendu, c'est à dire minimisent un résidu caractéristique de l'erreur. Ces méthodes sont détaillées dans les sections suivantes.

9.5.2 Méthodes de sous-espace

Les méthodes de réduction de modèle (voir section 3.3.2 et Refs. [41, 101]) conduisent directement à des méthodes d'expansion. Considérons un modèle réduit caractérisé par le sous-espace engendré par T et la projection (3.55) des équations du modèle initial. Si la taille du modèle réduit est égale au nombre de capteurs, on peut choisir une base \tilde{T} du sous-espace engendré par T telle que $[c][T] = [I]$. La relation de projection indique alors que la réponse en tout les DDLs du modèle complet est donnée par $\{y_{\text{ex}}\} = [\tilde{T}] \{y_{\text{id}}\}$.

De manière plus générale, on peut définir une interpolation de la forme

$$\{q_{\text{ex}}\} = [T] [c_{\text{id}}T]^+ \{c_{\text{q}}\} \quad (9.35)$$

où $[c_{\text{id}}]$ est la matrice d'observation caractérisant les capteurs expérimentaux pour le modèle analytique et $[c_{\text{id}}T]^+$ dénote un pseudo-inverse de $[c_{\text{id}}T]$.

Si le modèle réduit possède autant de vecteurs que de capteurs ($NR = NS$), il s'agit d'un inverse simple. Il y a dans ce cas une correspondance directe entre réduction et expansion puisque l'expansion correspond à la cinématique considérée pour la réduction. L'application de divers critères de comparaisons aux modèles réduits et complets ne donne cependant pas les mêmes résultats.

On notera aussi que la base $[\tilde{T}] = [T] [c_{\text{id}}T]^{-1}$ est une base de réduction telle que $[c_{\text{id}}\tilde{T}] = I$. Si $NR = NS$, l'expansion (9.35) est donc équivalente à la réduction sur un modèle où les sorties expérimentales sont utilisées comme DDLs.

Si $NR \leq NS$, on pourra utiliser

$$[c_{\text{id}}T]^+ = \left[[c_{\text{id}}T]^T [c_{\text{id}}T] \right]^{-1} [c_{\text{id}}T]^T \quad (9.36)$$

ou tout autre pseudo-inverse approprié (voir les discussions sur la SVD).

Pour les cas où $NR > NS$, il est indispensable de donner un sens physique au pseudo-inverse ce que l'on fait avec un critère d'erreur sur le résultat comme exposé dans la section 9.5.3.

Les bases d'expansion généralement considérées ici sont les modes normaux basse fréquence du modèle EF nominal (on parle alors d'**expansion modale**), les modes de contraintes associés aux DDL mesurés (on parle alors de d'**expansion statique**), les réponses harmoniques à des déplacements unitaires sur les capteurs (on parle alors de d'**expansion dynamique**).

L'utilisation d'une base de réduction de dimension inférieure au nombre de capteurs se justifie dans la mesure où les mesures expérimentales ne sont pas exactes et l'introduction d'un pseudo-inverse permet donc un certain lissage des solutions retenues.

9.5.3 Méthodes minimisant une erreur de modélisation

Les méthodes de sous-espace n'utilisent l'information du modèle que pour construire le sous-espace. Une classe plus générale de méthodes pose l'expansion comme un problème de minimisation combinant les erreurs de modélisation et d'essai.

Les erreurs d'essai sont prise en compte à l'aide de (9.34), alors que les erreurs de modélisation sont estimées par une norme de résidu dynamique (9.16). On combine ces deux erreurs en une somme pondérée conduisant à un problème de moindres carrés généralisé

$$\min_{q_{j,ex}} \|R(q_{j,ex})\|_K^2 + \gamma_j \epsilon_j \quad (9.37)$$

où il peut être utile de changer la pondération relative (coefficient γ_j) de manière à obtenir une valeur prédéterminée de l'erreur relative sur la mesure (9.8) (ce qui revient à poser un problème de minimisation avec inégalité quadratique comme dans [102]).

Ce type de formulation est beaucoup plus général que les méthodes de sous-espaces, il donne un mécanisme pour définir des projecteurs obliques [103] et a été l'objet de diverses recherches (voir entres autres [104, 102, 98]). Son application à des modèles industriels est cependant assez rare. Les méthodes de résolution sur base réduite proposées en [105] permettent des implémentations peu coûteuses ce qui devrait rendre ces approches plus populaires.

9.5.4 Reduction or expansion for correlation

A number of correlation criteria assume that test results are known on all DOFs of the FEM model. This can be achieved through reduction (use sensors as DOFs) or expansion (use a procedure to estimate test motion at DOFs).

For reduction, one seeks the solution of the FEM problem within a subspace of basis T . Assuming that this basis

- has as many vectors as there are sensors ($NR = NS$) and
- has independent vectors from the observation viewpoint (that is such that $[c][T]$ is invertible)

one can build a dynamic reduced model by projecting the model on the basis

$$\left[\tilde{T} \right] = [T] [cT]^{-1} \quad (9.38)$$

Les DDLs du modèle réduit correspondent clairement aux capteurs (puisque $c\tilde{T} = I$) ce qui permet d'utiliser les critères mécaniques évoqués précédemment (avec le modèle réduit (3.55)).

L'évaluation des critères sur des modèles réduits présentent les limitations suivantes.

Les conditions d'équilibre ou d'orthogonalité ne sont généralement pas censées être vérifiées exactement pour le modèle réduit. Il y a donc un biais systématique généralement mal maîtrisé.

Pour permettre une représentation correcte du comportement dynamique par un modèle réduit, l'analyse modale nous apprend qu'il faut tenir compte de la bande de fréquence considérée (NM nombre de modes à retenir) et du nombre d'entrées (NA nombre de corrections statiques à introduire). L'utilisation d'un modèle réduit ne peut donc être efficace que si $NS > NM + NA$ (si on a suffisamment de capteurs). Il faut par ailleurs vérifier que la méthode de réduction retenue est bien valable dans le cas considéré.

En recalage de modèle, les propriétés du modèle et donc les bases de réduction changent au cours des itérations. Le coût numérique est donc souvent très significatif.

Les méthodes d'expansion basiques, méthodes de sous-espaces, sont très similaires, voir strictement équivalentes, à des méthodes de réduction. Il n'y a alors pas vraiment de différence. Les méthodes d'expansion évoluées prennent en compte le critère de corrélation pour estimer le modèle étendu. Ces méthodes sont souvent coûteuses numériquement. Le juste milieu semble donc être l'utilisation de méthodes d'expansion sur base réduite.

9.6 Comparaisons entre fonctions de transfert

Comme pour les déformées, on distingue les approches utilisant le modèle élément fini et celles utilisant le modèle entrée/sortie.

Pour une FRF étendue ou un modèle réduit (avec $c = I$), le résidu dynamique est donné par

$$R(s, u(s)) = [Ms^2 + Cs + K] \{y_{\text{id}}(s)\} - [b] \{u_{\text{id}}(s)\} \quad (9.39)$$

qui permet d'étudier la corrélation comme indiqué en section 9.3.1. Cette approche a par exemple été utilisée pour corréler le comportement statique d'un pont [106] et "voir" l'influence de dommages.

Pour la comparaison entrée/sortie, on peut comparer directement les fonctions de transfert mesurées et calculées pour des paires d'entrées-sorties données. Dans la suite, les indices i et j dénotent le capteur et l'excitateur, respectivement, et Ω représente un ensemble de NF points de fréquences choisis pour la comparaison.

Le coût quadratique est souvent utilisé en identification ainsi qu'en recalage car il conduit à des problèmes de moindres carrés pour lesquels de nombreux solveurs numériques sont disponibles. Son autre avantage est de fournir de manière algébrique les gradients de la fonction coût, ce qui évite d'utiliser des formules d'approximation de type différences finies. La comparaison entre deux fonctions de transfert via un coût quadratique s'écrit

$$J_{ij}(\Omega) = \sum_{\{k \in \Omega\}} |H_{\text{id}ij}(s_k) - H_{ij}(s_k)|^2 \quad (9.40)$$

De même, on peut travailler avec un coût logarithmique qui est plus coûteux numériquement et ne conduit pas aux gradients aussi facilement, mais offre l'avantage d'être beaucoup plus sensible aux anti-résonances :

$$J_{ij}(\Omega) = \sum_{\{k \in \Omega\}} \left| \log \frac{H_{\text{id}ij}(s_k)}{H_{ij}(s_k)} \right|^2 \quad (9.41)$$

La sensibilité aux anti-résonances est une propriété importante car la qualité des modes propres identifiés est affectée autant par les résonances que par les anti-résonances. Le coût quadratique a tendance à "lisser" les différences aux anti-résonances car celles-ci correspondent à des déplacements locaux très petits. De manière intuitive, ces propriétés s'expliquent par le comportement au voisinage de zéro puisque :

$$\lim_{x \rightarrow 0} (x)^2 = 0 \qquad \lim_{x \rightarrow 0} \log(x) = -\infty \qquad (9.42)$$

De nombreux auteurs considèrent des coûts pondérés pour comparer des fonctions de transfert. Cependant, le choix de la pondération peut influencer significativement le résultat et l'implémentation de certaines pondérations est difficile. Dans le cas général, si les points des fonctions de transfert sont rangés sous forme d'un vecteur colonne :

$$\{H_{ij}(\Omega)\} = \begin{Bmatrix} H_{ij}(s_1) \\ H_{ij}(s_2) \\ \vdots \\ H_{ij}(s_{NF}) \end{Bmatrix} \qquad (9.43)$$

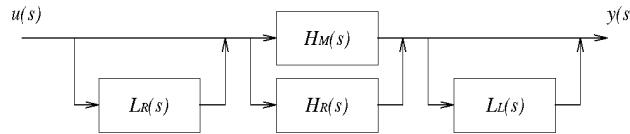
alors la différence pondérée entre fonctions de transfert s'écrit comme :

$$J_{ij}(\Omega) = (H_{\text{id}ij}(\Omega) - H_{ij}(\Omega))^H [W] (H_{\text{id}ij}(\Omega) - H_{ij}(\Omega)) \qquad (9.44)$$

où l'opérateur $()^H$ l'hermitien (conjugué transposé).

Pour mieux comprendre la motivation conduisant à l'utilisation de ces différents indicateurs, supposons qu'il existe un modèle linéaire exact $H_M(s)$. On peut représenter l'erreur du modèle courant sous forme de contributions additives et/ou multiplicatives telles que (voir figure ci-dessous pour une représentation en diagramme bloc)

$$[H(s)] = [I + L_L(s)] [H_M(s) + H_R(s)] [I + L_R(s)] \qquad (9.45)$$



L'utilisation d'erreurs additives et/ou multiplicatives bien qu'a priori arbitraire est souvent dictée par le type d'erreur représentée. Ainsi un bruit non corrélé sur la mesure de l'entrée, conduit pour l'estimateur H1 [24] (utilisé pour la détermination expérimentale de fonctions de transfert) à une erreur donnée sous la forme multiplicative

$$[H_1/H] = 1 / (1 + G_{nn}(s)/G_{uu}(s)) \qquad (9.46)$$

où G_{nn} est le spectre du bruit non corrélé et celui de l'entrée réelle.

De même, les effets liés au comportement non idéal d'un capteur (fonction de transfert non constante entre grandeur estimée et signal mesuré) ou aux délais temporels d'une chaîne de mesure sont généralement facilement représenté par une erreur multiplicative.

Les différentes représentations de l'erreur conduisent à des critères de comparaison de modèles. Pour des modèles caractérisés par la donnée de leur fonctions de transfert (matrice H avec autant de lignes que de points de fréquence et de colonnes que de fonctions de transfert). On a ainsi défini plus haut le critère quadratique lié à une représentation additive de l'erreur et le critère logarithmique lié à une représentation multiplicative.

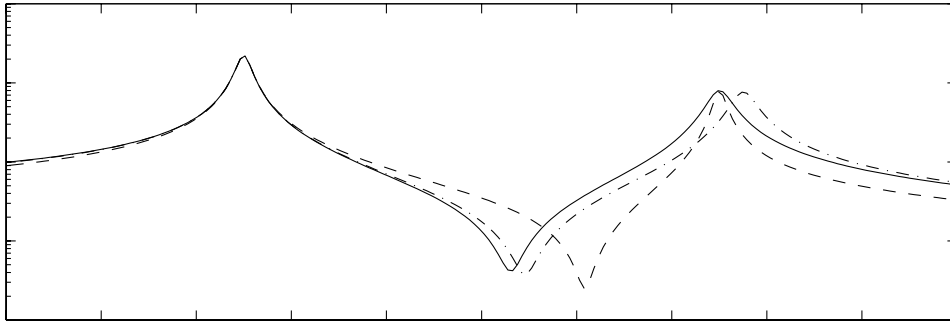


FIGURE 9.7 – Modèles non paramétriques comparables de trois fonctions de transfert. Modèle (—) 1, (---) 2, (-·-) 3

La différence entre deux modèles peut être représentée soit de manière additive soit de manière multiplicative. La mesure de cette différence par un critère associé à la représentation choisie peut varier de manière significative d'un critère à l'autre. Ainsi, les modèles 1 et 3 de la figure 9.7 sont, par rapport à la comparaison des modèles 1 et 2, plus éloignés d'un facteur 5 pour le critère additif et plus proches d'un facteur 5 pour le critère multiplicatif.

Le modèle 2 est assez typique d'une erreur d'identification à partir de données expérimentales. Pour un pôle dont l'amortissement est inexact, l'estimation du résidu est généralement biaisée de manière à conserver le niveau de réponse au niveau du maximum de réponse. Le modèle 3 est lui assez typique d'une erreur liée à un modèle élément fini. On a une erreur de 2% sur la fréquence mais l'estimation du résidu est correcte.

Pour une application en identification, le modèle 2 est donc qualitativement plus proche que le modèle 3 car, pour des structures faiblement amorties, il est facile de déterminer les fréquences expérimentales avec une très grande précision. Pour la comparaison d'un modèle élément fini avec des résultats expérimentaux, des erreurs de quelques pour cents sur les fréquences sont attendues alors que les résidus généralement basés sur le modèle non amorti ne peuvent compenser les erreurs du modèle d'amortissement. Le modèle 3 est donc qualitativement meilleur que le modèle 2.

Pour l'identification (détermination de paramètres modaux à partir de données expérimentales) le critère additif paraît donc plus adapté (voir applications [107, 77]), alors que pour la corrélation calculs/essais c'est le critère multiplicatif (voir par exemple [108]). Le critère additif étant souvent choisi pour ses propriétés mathématiques (c'est une forme quadratique), son utilisation dans le cadre de la corrélation calculs / essais demande de prendre en compte le fait que des erreurs de l'ordre du pour cent sur les fréquences sont faibles (par exemple en sélectionnant des fréquences éloignées des résonances [109]). On notera aussi que des critères combinant les réponses à plusieurs fréquences semblent aussi très prometteurs [110].

Finalement, on peut aussi comparer les FRFs en plusieurs capteurs mais au même points de fréquence en utilisant les critères de la section 9.2.

9.7 Conclusion

On a vu dans ce chapitre qu'il existe une grande variété de critères de corrélation calculs/essais.

Le premier critère devrait toujours être une **inspection visuelle** des déformées modales si possibles animées (on voit beaucoup plus de choses sur un écran que sur une feuille de papier imprimée).

Les mesures de corrélation MAC et COMAC sont très utiles car simples de mise en oeuvre mais peu précises et donc à manipuler avec beaucoup de précautions.

Les critères en relation d'équilibre et d'erreur en loi de comportement sont très utiles pour la localisation spatiale des erreurs.

Les comparaisons de fonctions de transfert et de flexibilités statiques sont des alternatives ouvertes et utiles dans un certain nombre d'applications.

Chapitre 10

Parameterization, sensitivities, reanalysis

10.1 Introduction

10.1.1 Motivation

In many applications, one is not interested in a single model but rather a family. For example, during optimization one will vary design parameters to achieve a better performance in some numerical sense. FEM updating is a particular case of optimization where the performance is the correlation with an experimental characterization of the structure. When considering **stochastic** models, one will seek to assess the **robustness** of the performance to changes that are characterized as stochastic parameters.

The first step of parametric studies is to introduce parametrization as discussed in section 10.1.2 . One then typically has two objectives : compute the first order derivatives of responses with respect to parameter changes (so called **sensitivities**, see section 10.2) ; make predictions at non nominal points using perturbation methods or the more accurate reanalysis (see discussion in section 10.4).

Sensitivities can be used in two ways : directly to help in analyzing the relative influence various parameters and indirectly in perturbation methods (using a first order Taylor expansion of the relation between parameters and responses). Perturbation methods have poor performance for predicting modes. Reanalysis based on reduced models is always more accurate and typically only marginally more costly. Model reduction should thus **always** be preferred. Thus the good reason to compute sensitivities is to analyze the influence of various parameters.

In this chapter, one will present usual methods for computing sensitivities of static responses, modeshapes, frequency response functions and integral objective functions, since these are often considered in updating problems.

10.1.2 FEM model parametrization

FEM models are described by a set of geometric and material properties. A parameterized model potentially lets all properties change (node positions, values characterizing the constitutive law, ...). Formally, one can define a vector of parameters in the model

$$\{p\} = \{p_0\} + \{\Delta p\} \quad (10.1)$$

In all cases, FEM software is able to link model parameters p to mass $M(p)$, stiffness $K(p)$ and possibly damping $C(p)s + iB(p)$ matrices.

Topology optimization algorithms available in multiple FEM environments consider element by element parametrizations with strategies to eliminate elements that do not contribute to a target response. Level set methods are also used to define boundaries in such methods [111].

In FEM updating, unknown parameters are typically grouped by area where a constant value is expected. It is then common to describe the global matrices as linear combinations, with variable coefficients but constant matrices

$$[K(p)] = \sum_{j=1}^{NP} \beta_j(p) [K_j] \quad (10.2)$$

Element matrices are obtained by numerical integration of a quadratic functions with coefficients depending linearly on constitutive parameters. The linear combination of matrices thus easily accounts for changes in moduli, density, shell thickness, beam sections, ...

For beams and shells section parameters can be considered as generalized constitutive parameters. Thus for shells, the element matrix can be written as a polynomial in thickness t for membrane, t^3 for bending and t^2 for coupling effects (found in composite or elements with off-sets). One thus considers three dependent coefficients $\beta_1 = t$, $\beta_2 = t^3$, $\beta_3 = t^2$. In many practical applications, shell elements mostly work in a single mode (bending or membrane). One then consider a single multiplicative coefficient β and then uses an approximate relation to recover the thickness. Thus, for bending dominated motion, the thickness associated with parameter β is linked to the initial thickness t_0 by

$$t^3 = \frac{\beta_0 t_0^3}{\beta} \quad (10.3)$$

In practice, one usually defines a coefficient for each element matrix (or sub-matrix when separating membrane and bending matrices for example). The linear combination thus becomes

$$[M(p)] = \sum_{j=1}^{NE} \alpha_k(p) [M_k^e] \quad [K(p)] = \sum_{j=1}^{NE} \beta_k(p) [K_k^e] \quad (10.4)$$

To determine the values of elementary coefficients, one groups them in elements sets (e_i, e_j) and associate them with mass p_i or stiffness parameters p_j . Thus for a mass parameter p_i linked to element set e_i , one has

$$\begin{aligned} \alpha_k, \beta_k &= 1 \quad \text{pour } k \notin e_i \text{ ou } e_j \\ \alpha_k &= p_i \quad \text{pour } k \in e_i \\ \beta_k &= p_j \quad \text{pour } k \in e_j \end{aligned} \quad (10.5)$$

Optimizing the reassembly is often a key step in model optimization and updating software. Strategies to avoid reassembly include polynomial developments of the dependence between p

and $K(p)$ and the matrix disassembly method [112] (where $K(p)$ is described as a product of the form $[C] \left[\backslash \beta(p) \backslash \right] [C]^T$).

The name **shape optimization** is used methods where functional definition of node changes are described by a restricted set of parameters. Such changes can be associated with CAD parameters or shapes described on the mesh (more related to morphing) [113]. The use of model reduction in this context is still an open topic.

For a given mesh, the properties of each element can theoretical vary independently. In practice there is strong coupling. Thus the thickness of a press formed part can be wrong, but the error depends on the forming process and the values of neighbouring elements are correlated. Similarly in soils, properties vary significantly but with variation sizes associated with the size of inclusions in the soil.

A realistic description of uncertainties thus uses a linear combination of uncertain variables fields $P(x)$ describing the local evolution of properties (behavior or geometry) as a function of space. One thus uses

$$p(x) = \sum_{k=1}^{NPI} P(x) \tilde{p} \quad (10.6)$$

The grouping of mass and stiffness parameters (10.5) corresponds to the choice of $P(x)$ fields equal to 1 on selected elements and zero elsewhere.

The building of $P(x)$ fields starting from considerations of correlation between properties has been largely addressed in the litterature on stochastic finite elements [114, 115]. Keywords to search for this litterature are *polynomial chaos* or *Karhunen-Loève theorem*.

The selection of independent parameters is a key difficulty of most applications. Procedures to perform that selection in the case of FEM updating are discussed in section 11.2.1.

10.2 Simple example : static responses

10.2.1 Exact sensitivities

Let us start with the simple case of sensitivity computations for static responses. For all parameters p , the static response is solution of

$$[K(p)] \{q\} = \{F\} \quad (10.7)$$

By differentiating this equation, one obtains

$$[K(p)] \left\{ \frac{\partial q}{\partial p_i} \right\} = \left(\left\{ \frac{\partial F}{\partial p_i} \right\} - \left[\frac{\partial K}{\partial p_i}(p) \right] \{q\} \right) \quad (10.8)$$

which is a new problem qui can be readily solved to obtain the sensitivities $\{\partial q / \partial p_i\}$. It useful to note that the stiffness matrix $K(p)$ can be kept and used to compute both the static response (10.7) and its sensitivities (10.8). Since the computations are efficient, computing the exact static sensivity is the best approach.

10.2.2 Finite difference approximation

Although the exact analytical solutions should be preferred whenever possible, understanding the properties of finite difference approximations is useful.

For a static problem of the form $[K(p)] \{q\} = \{F\}$, the simplest finite difference scheme is to compute the response at two nearby points $\{p\}$ and $\{p + \delta p_i\}$ of design space

$$[K(p)] \{q(p)\} = \{F\} \quad (10.9)$$

and

$$[K(p + \delta p_i)] \{q(p + \delta p_i)\} = \{F\} \quad (10.10)$$

then to use a two point approximation of the derivative

$$\left\{ \frac{\partial q}{\partial p_i} \right\} \approx \left\{ \frac{\delta q}{\delta p_i} \right\} = \frac{1}{\delta p_i} (\{q(p)\} - \{q(p + \delta p_i)\}) \quad (10.11)$$

One can proceed in a similar fashion for any quantity of interest (eigenvalues, FRFs, ...).

One of the major difficulties of finite difference methods is the *a priori* selection of the step size. A robust procedure would thus examine convergence for decreasing step sizes. In the case of static response sensitivities, one would thus use an algorithm of the form

For each design parameter, $i = 1 \cdots N_p$:

Iterate until convergence, $n = 1 \cdots$

- **Decrease step size** $\delta p_i^{(n)} \leftarrow \frac{\delta p_i^{(n-1)}}{10}$;
- **Assemble** $[K(p_i + \delta p_i^{(n)})]$;
- **Solve** $[K(p_i + \delta p_i^{(n)})] \{q\} = \{F\}$;
- **Estimate** $\left\{ \frac{\partial q}{\partial p_i} \right\}$;
- **Examine convergence.**

End of convergence loop for δp_i .

End of main loop.

Another difficulty is linked to the fact that predictions are often approximate. Given ϵ , the numerical precision on a quantity ψ whose derivative one seeks to estimate. The solutions obtained at two design points $\{p\}$ and $\{p + \delta p_i\}$ are known with precision ϵ . Thus

$$\tilde{\psi}(p) = \psi(p) + \epsilon(\psi), \quad \tilde{\psi}(p + \delta p_i) = \psi(p + \delta p_i) + \epsilon(\psi) \quad (10.12)$$

The partial derivative is thus the sum of the exact "estimate" of the derivative plus an error due to the numerical solution

$$\frac{\partial \psi}{\partial p} \approx \frac{\tilde{\psi}(p + \delta p_i) - \tilde{\psi}(p)}{\delta p_i} = \frac{\psi(p + \delta p_i) - \psi(p)}{\delta p_i} + \frac{\epsilon(\psi(p + \delta p_i)) - \epsilon(\psi(p))}{\delta p_i} \quad (10.13)$$

It appears that the second term can become significant when the increment in the objective $\psi(p + \delta p_i) - \psi(p)$ becomes similar in size to numerical precision $\epsilon(\psi)$ which propagates numerical errors. For modal frequencies, errors of the order of 10^{-5} are common. For modeshapes, the error can be significantly larger on some DOFs.

In conclusion attention must always be paid to convergence problems.

10.2.3 Perturbation and reanalysis

Historically, sensitivity computations have often been used for **perturbation methods**, where one approximates the solution of

$$[K(p_i + \delta p_i)] \{q(p_i + \delta p_i)\} = \{F(p_i + \delta p_i)\} \quad (10.14)$$

Assuming small perturbations

$$\frac{|\delta p_i|}{p_i} \ll \mathbf{1} \quad (10.15)$$

one can use a first order Taylor series expansion to express the solution of the perturbed system based on that of the initial system

$$\{q(p_i + \delta p_i)\} \approx \{q(p_i)\} + \left\{ \frac{\partial q_i}{\partial p_i} \right\} \delta p_i \quad (10.16)$$

where the sensitivity is computed using (10.8).

For more accurate predictions, it is strongly advisable to use the **reanalysis** of the **reduced model** obtained as follows. One seeks approximations of the form

$$\{\hat{q}(p_i + \delta p_i)\} = [T] \left[[T]^T [K(p_i + \delta p_i)] [T] \right]^{-1} \left\{ [T]^T \{F(p_i + \delta p_i)\} \right\} \quad (10.17)$$

within the subspace spanned by the initial static response and its sensitivity(ies)

$$[T] = \left[\{q(p_i)\} \quad \frac{\partial q}{\partial p} \right] \quad (10.18)$$

The additional numerical cost is generally minimal and the precision is guaranteed to be optimal (general property of Rayleigh Ritz methods). In the reduction basis, one can keep sensitivities to multiple parameters, modes associated with multiple values of p , etc.

10.2.4 Coûts quadratiques, état adjoint

Le calcul de sensibilité est généralement utilisé dans le cadre d'une **optimisation** où l'on cherche à minimiser une fonction objectif $J(p, q)$. Pour l'implémentation d'un solveur numérique, il faut pouvoir accéder aux dérivées partielles de la fonction coût. La différenciation du critère d'optimisation donne

$$\begin{aligned}
\frac{\partial J}{\partial p_i} &= \frac{\partial J}{\partial p_i}(p; q) + \left\{ \frac{\partial J}{\partial q}(p; q) \right\}^T \left\{ \frac{\partial q}{\partial p_i} \right\} \\
&= \frac{\partial J}{\partial p_i}(p; q) + \left\{ \frac{\partial J}{\partial q}(p; q) \right\}^T [K(p)]^{-1} \left(\left\{ \frac{\partial f}{\partial p_i} \right\} - \left[\frac{\partial K}{\partial p_i}(p) \right] \{q\} \right)
\end{aligned} \tag{10.19}$$

Il n'est pas interdit de remarquer que K étant symétrique, on peut réécrire l'expression précédente sous la forme

$$\frac{\partial J}{\partial p_i} = \frac{\partial J}{\partial p_i}(p; q) + \left(\left\{ \frac{\partial f}{\partial p_i} \right\} - \left[\frac{\partial K}{\partial p_i}(p) \right] \{q\} \right)^T [K(p)]^{-1} \left\{ \frac{\partial J}{\partial q}(p; q) \right\} \tag{10.20}$$

où l'on calcule maintenant la réponse statique à la sollicitation $\left\{ \frac{\partial J}{\partial q}(p, q) \right\}$ que l'on projette dans la direction des pseudo-charges $\left(\left\{ \frac{\partial f}{\partial p_j} \right\} - \left[\frac{\partial K}{\partial p_j}(p) \right] \{q\} \right)$. Cette utilisation de la dualité charge déplacement peut être justifiée par l'introduction d'un **état adjoint**. Mais l'idée essentielle est de limiter le nombre de calculs statiques nécessaires. Pour un nombre significatif de paramètres, la dimension de $\left\{ \frac{\partial J}{\partial q}(p, q) \right\}$ (nombre de déplacements/contraintes optimisés) est inférieure à celle de $\left(\left\{ \frac{\partial f}{\partial p_j} \right\} - \left[\frac{\partial K}{\partial p_j}(p) \right] \{q\} \right)$ (nombre de paramètres).

En pratique le coût utilisé est souvent la norme quadratique d'une fonction linéaire des DDL : on cherche à minimiser la flèche d'un pont ou la contrainte en un point de la structure. Dans le premier cas, on sait que les déplacement physiques et les DDLs sont reliés par la relation d'observation $\{y\} = [c] \{q\}$. Pour le cas des contraintes, c'est un peu moins évident mais les déformations sont des fonctions linéaires des déplacement (hypothèse des petites déformations) et en élasticité linéaire les contraintes sont des fonctions linéaires des déformations. On peut donc aussi écrire pour les contraintes une relation de la forme $\sigma = [c_\sigma] \{q\}$. On notera cependant que les contraintes principales n'étant pas une fonction linéaire du tenseur des contraintes, il n'y a pas de relation d'observation linéaire pour ces valeurs. Dans de tels cas, on a

$$\frac{\partial J}{\partial p_i} = \left\{ \frac{\partial J}{\partial q}(p; q) \right\}^T \left\{ \frac{\partial q}{\partial p_i} \right\} = \{q\}^T [c^T c] \frac{\partial q}{\partial p_i} \tag{10.21}$$

10.3 Frequencies and mode shapes

10.3.1 Exact solution

Modes are solution of the eigenvalue problem

$$[K(p) - \omega_j^2 M(p)] \{\phi_j\} = [Z(\omega_j, p)] \{\phi_j\} = \{0\} \tag{10.22}$$

and verify two orthogonality conditions with respect to mass

$$\{\phi_k\}^T [M] \{\phi_j\} = \delta_{jk} \tag{10.23}$$

and stiffness

$$\{\phi_k\}^T [K] \{\phi_j\} = \omega_j^2 \delta_{jk} \quad (10.24)$$

The mass normalization of mode j , linked to the constant $\{\phi_j\}^T [M] \{\phi_j\}$, is arbitrary and will be assumed to be equal to 1 in all cases (as shown in the orthogonality conditions (10.23)-(10.24)).

Equation (10.22) being valid for all values of p , its derivative with respect to p is also equal to zero, which one easily shows to result in

$$[Z(\omega_j)] \left\{ \frac{\partial \{\phi_j\}}{\partial p} \right\} = \{B(\omega_j)\} \quad (10.25)$$

where

$$B(\omega_j) = \left[-\frac{\partial K}{\partial p} + \frac{\partial \omega_j^2}{\partial p} M + \omega_j^2 \frac{\partial M}{\partial p} \right] \{\phi_j\} \quad (10.26)$$

By definition of modes (10.22), the dynamic stiffness $[Z(\omega)]$ is singular at modal frequencies ω_j so that that equation (10.25) does not necessarily have a solution. The kernel of $[Z(\omega)]$ is however $\{\phi_j\}$ and it is a well known theorem of linear algebra that equations of the form $Zq = B$ with Z singular have solutions if and only if B is orthogonal to the kernel of Z^T . Thus here, one must have $\{\phi_j\}^T B(\omega_j) = 0$ which defines the sensitivity of modal frequencies

$$\frac{\partial \omega_j^2}{\partial p} = \{\phi_j\}^T \left[\frac{\partial K}{\partial p} - \omega_j^2 \frac{\partial M}{\partial p} \right] \{\phi_j\} \quad (10.27)$$

Again, it is known from linear algebra that solutions of (10.25) take the general form $\partial \{\phi_j\} / \partial p = \psi_j + \alpha \phi_j$ where ψ_j is an arbitrary particular solution of $[Z(\omega_j)] \{\psi_j\} = \{B(\omega_j)\}$.

As proposed by Nelson [116], a particular solution of (10.25) can be determined by imposing that one of the components of ψ_j to be equal to zero. This particular solution clearly exists as long as the corresponding component of ϕ_j is non-zero. Knowing that a component of ψ_j is zero, one can eliminate a row and a column of (10.25) which leads to a non-singular set of equations that can be solved relatively easily. This solution however requires the factorization of a block of $Z(\omega_j)$. This factorization must be performed at the frequency of each of the desired modeshape sensitivities which tends to be prohibitively expensive for realistic finite element models.

Finally, a condition is needed to define the coefficient α in the general form of the solution. Assuming that the modeshape is always mass normalized as shown in (10.23), this condition can be derived with respect to p which leads to

$$\{\phi_j\}^T M \frac{\partial \{\phi_j\}}{\partial p} = -\frac{1}{2} \{\phi_j\}^T \frac{\partial M}{\partial p} \{\phi_j\} \quad (10.28)$$

Thus, given ψ_j a particular solution of $Z(\omega_j)\psi_j = B(\omega_j)$, the sensitivity of the mass normalized modeshapes is given by

$$\frac{\partial \{\phi_j\}}{\partial p} = \{\psi_j\} - (\phi_j^T M \psi_j + \frac{1}{2} \phi_j^T \frac{\partial M}{\partial p} \phi_j) \{\phi_j\} \quad (10.29)$$

Note that for cases with multiple modes, this discussion needs further considerations as found in section 10.3.4 and Refs. [117, 118, 119].

10.3.2 Approximations by projection of the solution

Projection methods are widely used to seek approximations of the properties of dynamic systems. The simplest of such approximations is the projection on a truncated modal basis. Condensation [40], component mode synthesis and substructuring methods [41], approximations on series of Krylov or Lanczos vectors [31] are other well known methods (the later are typically used to seek approximations of low frequency eigenvalues [16]).

All these methods are linked to the assumption that an accurate approximation of the response can be found in a subspace spanned by the columns of a rectangular projection matrix T (with N rows and $NR \ll N$ columns). As analyzed in Ref. [120], a constant basis can be used to approximate the solutions of a family of models characterized by the parameters p . The approximate modes of a model projected on the basis T are given by $\phi_j = T\phi_{jR}$ with ϕ_{jR} solution of the projected (reduced) eigenvalue problem

$$[T]^T [K(p) - \omega_{jR}^2 M(p)] [T] \{\phi_{jR}\} = \{0\} \quad (10.30)$$

Assuming that T is fixed, equation(10.30) is valid for all values of p and can be derived as done for the full order model in the previous section. The general form of the approximate sensitivity is clearly given by

$$\frac{\partial \{\phi_j\}_R}{\partial p} = [T] (\psi_{jR} + \alpha \phi_{jR}) \quad (10.31)$$

where ψ_{jR} is solution of

$$[T^T Z(\omega_j) T] \{\psi_{jR}\} = T^T \left[-\frac{\partial K}{\partial p} + \frac{\partial \omega_j^2}{\partial p} M + \omega_j^2 \frac{\partial M}{\partial p} \right] T \{\phi_{jR}\} \quad (10.32)$$

The method proposed by Fox and Kapoor [121] is the most widespread projection method used to approximate modeshape sensitivities. At any design point p , the projection basis is taken to be a truncated set of the exact modes at this design point $T = [\phi_1(p) \dots \phi_{NR}(p)]$. By multiplying equation (10.25) on the left by T^T and using the orthogonality conditions (10.23)-(10.24), one classically shows that a particular solution of the projected equation is given by

$$\psi_{jR} = \sum_{k \neq j} \frac{\{\phi_k\}^T \left[\frac{\partial K}{\partial p} - \omega_j^2 \frac{\partial M}{\partial p} \right] \{\phi_j\}}{\omega_j^2 - \omega_k^2} \{\phi_k\} \quad (10.33)$$

and that the component of the exact solution in the direction of ϕ_j is given by $\alpha = -\phi_j^T (\partial M \partial p) \phi_j / 2$.

Rather than using these expressions that are only valid for the exact modes at the current design point, one should realize that they correspond to the use of Nelson's exact method applied on the model projected on the associated basis. Nelson's method being applicable to any reduced sensitivity equation of the form (10.32) finding explicit expressions for the particular solution is not useful. Furthermore the cost associated to this evaluation is negligible since the dimension of the subspace (number of columns of T) is small compared to the size of the initial model.

When designing improved methods for the approximation of sensitivities, the effort should thus concentrate on building a reduction basis that will give accurate predictions of the sensitivities.

The truncated modal basis, while generally available is not the most efficient reduction basis in terms of allowing accurate predictions of modeshape sensitivities. As a first example of alternate bases, Ojalvo and Wang [122] realized that the estimates of $\phi_j(p)$ are often determined by projection of the model on a basis of Lanczos vectors which span the same subspace as the Krylov vectors given by $T_p = (K^{-1}M)^{p-1}K^{-1}T_0$ and thus proposed a method allowing the use of the same basis of Lanczos vectors for the estimation of modeshapes and modeshape sensitivities.

This approach continues to accept the cost of computing the exact modes at each iteration but uses a larger projection basis to estimate sensitivities, so that results can be expected to be more accurate. In practice robust results are obtained when some knowledge of the modification is taken into account as shown in the following sections.

10.3.3 Iterative determination of the exact solution

A first category of methods continues to accept the cost of computing the exact modes at each iteration but seeks to find a way of approximating the exact sensitivity. Such improvements will be found by complementing the modal basis of Fox's method or the Lanczos basis of Ojalvo's method by additional vectors that take properties of the modification into account.

Realizing that in the sensitivity equation (10.25), the second member $B(\omega_j)$ corresponds to a load that is representative of the modification, it is useful to complement the basis T by the static responses to this load

$$T_C = \left[\{ \phi_j(p) \} \quad \left[\hat{K} \right]^{-1} [B(\omega_1, p) \dots B(\omega_n, p)] \right] \quad (10.34)$$

As shown here, the augmented basis should include the static responses to modification loads of several modes (the one wishes to compute the sensitivity of) rather than, as proposed in Ref. [123], consider a static correction for each mode. In presence of rigid body modes, $\left[\hat{K} \right]^{-1}$ should be a pseudo-inverse of the stiffness (see section 5.3.2 for a discussion of the problem).

This first level correction can be extended using the following vectors of the Krylov series $T_k = \left[\tilde{K}^{-1}M \right]^k \left[\tilde{K} \right]^{-1} B_{1\dots n}$ but care must be taken to orthogonalize the successive additions T_k to the base subspace spanned by T_C . One could for example use the Lanczos orthogonalization scheme for this purpose. This would go back to the idea proposed by Ojalvo and Wang [122] but use $\left[\tilde{K} \right]^{-1} [B(\omega_1, p) \dots B(\omega_n, p)]$ as the base vectors to restart a block Lanczos algorithm.

Note that one saves a lot of time by using the same factorization of K to compute eigenvectors (using a subspace or Lanczos method) and the corrections (10.34) needed to compute accurate sensitivities.

10.3.4 Cas des fréquences multiples

On suppose maintenant que ω_j^2 possède une multiplicité NM strictement supérieure à 1. Ce cas est plutôt pathologique et essentiellement rencontré pour les modèles de structures ayant des symétries particulières. En particulier, les modèles axisymétriques possèdent des

fréquences doubles. Expérimentalement, il est impossible de construire une structure possédant des modes vraiment multiples mais la notion de modes très proches par rapport à leur niveau d'amortissement est importante.

Pour une valeur propre multiple, il existe donc un sous-espace modal de dimension NM . Tout vecteur du sous-espace est un mode propre. On peut donc choisir une base notée Φ , construite orthogonale en masse, c'est à dire telle que $\Phi^T M \Phi = I$, et dire que tout mode propre du sous-espace considéré est de la forme

$$[\phi_j] = \sum_{k=1}^{NM} c_{kj} \{\Phi_k\} = [\Phi_1 \ \dots \ \Phi_{NM}] \{c_j\} \quad (10.35)$$

La condition d'existence d'une sensibilité solution de (10.25) est toujours l'orthogonalité du second membre par rapport au noyau de Z dont Φ est une base. On a donc l'équation vectorielle $\Phi^T B(\omega_j) \{c_j\} = 0$ que l'on peut réécrire sous la forme du problème aux valeurs propres en $\{c_j\}$ et $\partial\omega_j^2/\partial p$

$$\left[\{\Phi\}^T \left[\frac{\partial K}{\partial p} - \omega_j^2 \frac{\partial M}{\partial p} \right] \{\Phi\} \right] \{c_j\} = \left[\{\Phi\}^T M \{\Phi\} \right] \{c_j\} \frac{\partial\omega_j^2}{\partial p} \quad (10.36)$$

qui définit, dans le sous-espace propre Φ , la direction c_j des "futurs" vecteurs propres de la structure perturbée et la sensibilité de la fréquence propre associée.

La sensibilité des déformées modales est toujours de la forme $\partial\{\phi_j\}/\partial p = \psi_j + [\Phi] \{c_j\} \alpha_j$ où ψ_j est une solution particulière arbitraire de $[Z(\omega_j)] \{\psi_j\} = \{B(\omega_j)\} \{c_j\}$.

La solution particulière peut être déterminée comme dans le cas des fréquences uniques en imposant une valeur nulle sur NM degrés de liberté qui ne s'annulent pas pour $\phi_j = [\Phi] \{c_j\}$.

L'indétermination sur le vecteur de constantes $\{\alpha_j\}$ se détermine comme précédemment à partir de la condition d'orthogonalité en masse (10.23) et des $\{c_j\}$ (obtenus en même temps que les sensibilités des valeurs propres en (10.36)).

10.4 Non nominal predictions and reanalysis

10.4.1 Perturbation vs. reanalysis

L'utilisation des sensibilités modales en calcul de perturbation (estimation de la réponse en $p + \Delta p$) n'est **pas conseillée** car peu précise. Pour les fréquences, le développement limité à l'ordre 1

$$\omega_j^2(p + \Delta p) \approx \omega_j^2(p) + \frac{\partial\omega_j^2}{\partial\{p\}} \{\Delta p\} + o(p) \quad (10.37)$$

est basé sur l'hypothèse que les déformées modales ne changent pas. Or cette hypothèse n'est généralement valable que sur un espace paramétrique très restreint.

Pour une estimation peu coûteuse des modes en un point différent du point courant, il est donc fortement conseillé d'utiliser le calcul exact des modes d'un modèle réduit (10.30). Parmi les nombreux choix possibles de base de réduction, on pourra considérer l'approche **multi-modèle** où l'on considère les modes exacts aux bornes d'une région paramétrique

$$T_M = [\{ \phi_j(p_1) \} \quad \{ \phi_j(p_2) \}] \quad (10.38)$$

l'approche **nominal plus sensibilité** où l'on conserve les modes exacts et leurs sensibilités aux point nominal (on remplacera généralement ce sous-espace par (10.34) utilisé pour calculer les sensibilités)

$$T_S = \left[\{ \phi_j(p_1) \} \quad \left. \frac{\partial \{ \phi_j \}}{\partial p} \right|_{p_1} \right] \quad (10.39)$$

La motivation des ces approches est que le modèle réduit ainsi créé permet de calculer une approximation des modes avec

$$[T^T K(p)T] - \omega_{jR}^2 [T^T M(p)T] \{ \phi_{jR} \} = \{ 0 \} \quad (10.40)$$

et de leurs sensibilités avec (10.31) avec un coût numérique minime et une assez bonne précision. Le choix de la méthode dépend du problème posé. Le multi-modèle est adapté à la recherche sur un intervalle connu alors que l'utilisation des sensibilité est plus approprié si on ne connais pas a priori la taille du pas.

Les limites des approches de réduction sont liées à l'augmentation du nombre de paramètres qui peuvent conduire à des modèles de trop grande taille et donc moins intéressants en coût de calcul.

10.4.2 Standard bases for reanalysis

Les techniques de réanalyse [120, 124] cherchent une solution approchée dans un sous-espace T indépendant de p en résolvant pour chaque valeur de p

$$[[T^T K(p)T] - \omega_{jR}^2(p) [T^T M(p)T]] \{ \phi_{jR}(p) \} = \{ 0 \} \quad (10.41)$$

Dans la mesure où $Z(p)$ est un polynôme matriciel en p , il est possible de réaliser la projection de chaque matrice une fois pour toute et donc une résolution très rapide de (10.41). La restitution des modes sur l'ensemble des DDLs est donnée simplement par $\{ \phi_j \} = [T] \{ \phi_{jR} \}$.

La question fondamentale est la construction d'une base T donnant de bonnes prédictions pour toutes les valeurs de p souhaitées. Quelques solutions classiques s'imposent

- la base nominale des modes $T = [\phi(p_0)_{1:NM}]$ [124],
- l'inclusion des sensibilités [120] (pour leur calcul on se reportera à [125, 126]),

$$T = \left[\phi(p_o) \quad \frac{\partial \phi}{\partial p_i} \right] \quad (10.42)$$

- l'approche multi-modèle [120] qui consiste à conserver les modes calculés en un certain nombre de points p_i

$$T = [\phi(p_1) \quad \phi(p_{NE})] \quad (10.43)$$

Pour une base de modes nominaux, les prédictions sont valables sur une zone de faible étendue car la base ne tient pas compte des variations du modèle [120]. En incluant les sensibilités, on obtient une zone de validité significativement plus grande mais dont la largeur n'est pas

facilement contrôlée. Pour une base multi-modèle, les prédictions de mode sont exactes pour les valeurs $p_i, i = 1 : NE$ et assez précises entre ces points. Par la suite on mettra en oeuvre une technique où sont conservés tous les modes d'un point central et les modes cibles en un certain nombre d'autres points.

Le modèle de méconnaissance à propager étant défini par un hypercube $p_i \in [p_i^{\min}, p_i^{\max}]$, il convient d'assurer la précision des prédictions sur l'ensemble des valeurs correspondantes. Les choix possibles de points conservés seront explicités dans la section suivante.

10.5 Response sensitivities

10.5.1 Transfer function sensitivities

Les fonctions de transfert pour une réponse en déplacement sont données par

$$\{y(s)\} = [c] [Ms^2 + Cs + K]^{-1} [b] \{u(s)\} \quad (10.44)$$

L'approche brutale est de calculer la sensibilité de la flexibilité dynamique $Z(s)^{-1} = [Ms^2 + Cs + K]^{-1}$. On utilise pour ceci la règle de calcul de la sensibilité de l'inverse d'une matrice

$$\frac{\partial}{\partial p_i} \{ [Z] [Z]^{-1} = [I] \} \quad \longrightarrow \quad \left[\frac{\partial Z^{-1}}{\partial p_i} \right] = -[Z]^{-1} \left[\frac{\partial Z}{\partial p_i} \right] [Z]^{-1} \quad (10.45)$$

Si cette expression est utilisée, on devra inverser $Z(s)$ à chaque point de fréquence considéré. Pour un modèle un tant soit peu réaliste (c'est à dire avec plusieurs milliers de DDL et dizaines de points de fréquences), ceci n'est pas raisonnable.

L'idée naturelle est alors de passer par une décomposition modale et d'écrire

$$\frac{\partial [H]}{\partial p} = [c] \sum_{j=1}^N \frac{\partial}{\partial p} \left(\frac{\{\phi_j\} \{\phi_j\}^T}{s^2 + \omega_j^2} \right) [b] \quad (10.46)$$

Mais comme toujours, il n'est pas possible de calculer tous les modes et il faut passer par une troncature modale qui doit presque toujours inclure une correction statique pour être d'une précision acceptable. En supposant le modèle réduit sur une base T indépendante de p , le calcul de la sensibilité de l'inverse appliqué à la flexibilité dynamique réduite conduit à

$$\{y(s)\} = [cT] [Z_R]^{-1} \left[[T]^T \left[\frac{\partial Z}{\partial p_i} \right] [T] \right] [Z_R]^{-1} [T^T b] \{u(s)\} \quad (10.47)$$

où $[Z_R(s)] = [T]^T [Z(s)] [T]$. Toutes les bases considérées dans le cours de réduction sont acceptables. Une base raisonnable conserverait les modes nominaux, leurs sensibilités, la correction statique aux entrées considérées, éventuellement la sensibilité de cette correction statique

$$[T] = \left[\begin{array}{c} [\phi_{1:NM}] \quad \left[\frac{\partial \phi}{\partial p} \right] \quad [\hat{K}]^{-1} [b] \quad \frac{\partial [\hat{K}^{-1} b]}{\partial p} \end{array} \right] \quad (10.48)$$

10.5.2 Sensibilité de critères intégraux

Lorsque l'on étudie une réponse transitoire, il est courant de considérer des critères de conception sous forme intégrale. Par exemple, on souhaite que la quantité d'énergie de déformation stockée durant un intervalle de temps $[t_i; t_f]$ ne dépasse pas une certaine valeur. Autre exemple, on souhaite contrôler la valeur moyenne des contraintes d'un système soumis à des cycles de chargement/déchargement. Le critère de conception est formulé de manière générale comme

$$J(p) = \int_{t_i}^{t_f} \mathcal{C}(p; t; x) dt \quad (10.49)$$

où la quantité scalaire \mathcal{C} dépend de la solution $\{x\}$ des équations du mouvement que l'on peut de manière générale mettre sous la forme d'un modèle d'état (voir le chapitre 2 pour plus de détails)

$$\{\dot{x}\} = [A] \{x\} + [B] \{u(t)\} \quad (10.50)$$

On souhaite estimer la sensibilité de ce critère par rapport aux variables de conception p_i . L'approche directe est résumée ainsi que la méthode de l'état adjoint qui permet de s'affranchir des difficultés rencontrées lors de l'implémentation de la méthode directe.

10.5.3 Approche directe

En supposant que l'intégrale converge et que les fonctions manipulées sont suffisamment régulières pour que l'on puisse intervertir l'intégration et la dérivation, la sensibilité du critère de conception par rapport à l'une des variables p_i s'écrit comme

$$\frac{dJ}{dp_i}(p) = \int_{t_i}^{t_f} \frac{d\mathcal{C}}{dp_i}(p; t; x) dt \quad (10.51)$$

La variable temps t est indépendante des paramètres de conception p_i , la sensibilité de la fonction scalaire \mathcal{C} est égale à

$$\frac{d\mathcal{C}}{dp_i} = \frac{\partial \mathcal{C}}{\partial p_i} + \left\{ \frac{\partial \mathcal{C}}{\partial x} \right\}^T \left\{ \frac{\partial x}{\partial p_i} \right\} \quad (10.52)$$

L'évaluation du gradient du critère \mathcal{C} par rapport au vecteur des états $\{x\}$ peut être une étape très coûteuse si la taille de $\{x\}$ est importante. De plus, si la dépendance de \mathcal{C} en $\{x\}$ n'est pas explicite, des différences finies doivent être utilisées, ce qui détériore d'autant la précision des calculs.

Enfin et surtout, le calcul de la sensibilité de $\{x\}$ par rapport aux paramètres de conception p_i nécessite la résolution d'une équation différentielle obtenue en dérivant l'équation du mouvement (10.50)

$$\frac{\partial}{\partial t} \left\{ \frac{\partial x}{\partial p_i} \right\} = [A] \left\{ \frac{\partial x}{\partial p_i} \right\} + \left[\frac{\partial A}{\partial p_i} \right] \{x\} + \left[\frac{\partial B}{\partial p_i} \right] \{u(t)\} + [B] \left\{ \frac{\partial u}{\partial p_i} \right\} \quad (10.53)$$

pour chaque paramètre p_i , $i = 1 \dots NP$. Cette étape apporte en général un lourd tribut au coût numérique de la méthode. Ces trois problèmes constituent généralement l'approche directe inabordable d'un point de vue numérique.

10.5.4 Méthode de l'état adjoint

Ici, les opérations coûteuses (et que l'on souhaiterait éliminer) portent sur l'évaluation des sensibilités $\left\{\frac{\partial \mathcal{C}}{\partial x}\right\}$ et surtout sur $\left\{\frac{\partial x}{\partial p_i}\right\}$.

La **méthode de l'état adjoint** consiste à introduire un multiplicateur de Lagrange $\{\lambda\}$ associé à (10.53) et appelé **état adjoint du système**. La formulation intégrale avec multiplicateur pour la vérification de (10.53) est donnée par

$$\int_{t_i}^{t_f} \{\lambda\}^T \left(\frac{\partial^2 x}{\partial t \partial p_i} - A \frac{\partial x}{\partial p_i} \right) dt = \int_{t_i}^{t_f} \{\lambda\}^T \left(\frac{\partial A}{\partial p_i} \{x\} + \frac{\partial B}{\partial p_i} \{u\} + [B] \frac{\partial \{u\}}{\partial p_i} \right) dt \quad (10.54)$$

Une intégration par parties du membre de gauche permet de remplacer la dérivation par rapport au temps de la solution $\{x\}$ en faisant toutefois apparaître un terme de dérivation de l'état adjoint par rapport au temps. Il vient

$$\begin{aligned} \left[\{\lambda(t)\}^T \left\{ \frac{\partial \{x\}}{\partial p_i} \right\} \right]_{t_i}^{t_f} &- \int_{t_i}^{t_f} \left(\left\{ \frac{\partial \lambda}{\partial t} \right\}^T - \{\lambda\}^T [A] \right) \left\{ \frac{\partial \{x\}}{\partial p_i} \right\} dt \\ &= \int_{t_i}^{t_f} \{\lambda\}^T \left(\frac{\partial A}{\partial p_i} \{x\} + \frac{\partial B}{\partial p_i} \{u\} + [B] \frac{\partial \{u\}}{\partial p_i} \right) dt \end{aligned} \quad (10.55)$$

On peut alors définir l'état adjoint $\{\lambda\}$ comme étant égal à la solution de l'équation différentielle (dite adjointe)

$$\left\{ \frac{\partial \lambda}{\partial t} \right\} - [A]^T \{\lambda\} = \left\{ \frac{\partial \mathcal{C}}{\partial x} \right\} \quad (10.56)$$

avec des conditions initiales qui permettent d'éliminer le terme de bord (c'est à dire le premier terme du membre de gauche) dans l'équation (10.55)

$$\{\lambda(t_i)\} = \{\lambda(t_f)\} = \{0\} \quad (10.57)$$

Cette définition de l'état adjoint nécessite toujours le calcul du gradient $\left\{\frac{\partial \mathcal{C}}{\partial x}\right\}$. La méthode de l'état adjoint est donc particulièrement intéressante lorsque ce calcul peut être réalisé simplement. Par exemple, si le critère représente une mesure des contraintes telle que

$$\mathcal{C} = \{\sigma(x(t))\}^T \{\sigma(x(t))\} \quad \text{avec} \quad \{\sigma(x)\} = [c] \{x\} \quad (10.58)$$

où la matrice $[c]$ représente la relation entre contraintes et déplacements nodaux, alors le gradient recherché peut être calculé de manière analytique comme :

$$\left\{ \frac{\partial \mathcal{C}}{\partial x} \right\} = 2 \left([c]^T [c] \right) \{x\} \quad (10.59)$$

Avec la définition (10.56)-(10.57) de l'état adjoint, il est maintenant possible de simplifier fortement l'expression de la sensibilité du critère intégral de conception. Celle-ci s'écrit en effet

$$\frac{dJ}{dp_i}(p) = \int_{t_i}^{t_f} \frac{d\mathcal{C}}{dp_i} dt = \int_{t_i}^{t_f} \left(\frac{\partial \mathcal{C}}{\partial p_i} + \left\{ \frac{\partial \mathcal{C}}{\partial x} \right\}^T \left\{ \frac{\partial x}{\partial p_i} \right\} \right) dt \quad (10.60)$$

Or, le terme gradient $\left\{ \frac{\partial \mathcal{C}}{\partial x} \right\}$ peut être remplacé par le membre de gauche de l'équation (10.56). Puis, le terme intégral résultant peut, à son tour, être remplacé par le membre de droite de l'équation (10.55). Sans calculs, on obtient

$$\frac{dJ}{dp_i}(p) = \int_{t_i}^{t_f} \frac{\partial \mathcal{C}}{\partial p_i} dt - \int_{t_i}^{t_f} \{\lambda\}^T \left(\frac{\partial A}{\partial p_i} \{x\} + \frac{\partial B}{\partial p_i} \{u\} + [B] \frac{\partial \{u\}}{\partial p_i} \right) dt \quad (10.61)$$

Cette écriture suppose toutefois que l'on calcule l'état adjoint en résolvant les équations (10.56)-(10.57). C'est le prix à payer pour obtenir cette énorme simplification.

10.6 Conclusions

Difficile de conclure sur cet ensemble d'outils essentiels pour d'autres applications plus complexes. On retiendra quand même qu'il vaut mieux

- **éviter les différences finies** tant qu'il est possible d'obtenir une dérivation analytique (même si celle-ci est assez complexe comme pour le cas des critères intégraux).
- **éviter la méthode des perturbations** et lui préférer l'utilisation d'une méthode de réduction à peine plus coûteuse en calcul mais qui a presque toujours la garantie d'être plus précise et est souvent la seule efficace pour des modifications non infinitésimales.
- penser, pour les dérivations de sensibilités analytiques, aux formulations intégrales qui introduisent des **états adjoints** comme intermédiaires de calculs.
- **soigner l'implémentation numérique** car les calculs de sensibilité sont centraux à de nombreux algorithmes d'optimisation et leur coût devient souvent prohibitif.

Chapitre 11

Finite element model updating

11.1 Introduction

FEM model updating is part of the model validation and verification process.

The *verification* phase deals with the check that the numerical methods lead to a solution that corresponds to the *conceptual numerical model* of the structure. The *validation* phase checks that the numerical model gives predictions that match experimental results.

Thus updating is part of the *validation* phase and deals with the search of parameters that minimize the distance between test and analysis results.

Updating can be decomposed in two steps : the definition of the family of possible models and the search for the best test/analysis match within that family. If the family is defined in terms of a parametric uncertainty model, then *updating* can be considered as an uncertainty reduction phase.

Historically, updating first considered matrix coefficients as parameters. The resulting **global methods** thus optimize changes of the form ΔM , ΔK with multiple constraints. These methods are summarized in section 11.4 but their industrial applications are few, and the author's opinion is that they should be classified as identification methods since the knowledge of geometrical or material properties is mostly unused.

In the second class of updating methods, called *local methods*, one seeks to optimize *physical parameters* that characterize the model geometry or constitutive laws. These methods first parametrize the model as will be detailed in section 11.2 then search for an optimal model within a given family as detailed in sections 11.4, 11.5.

Updating uses a number of methods, shown in figure 11.1, and detailed in earlier chapters. Many of these methods are interesting in themselves.

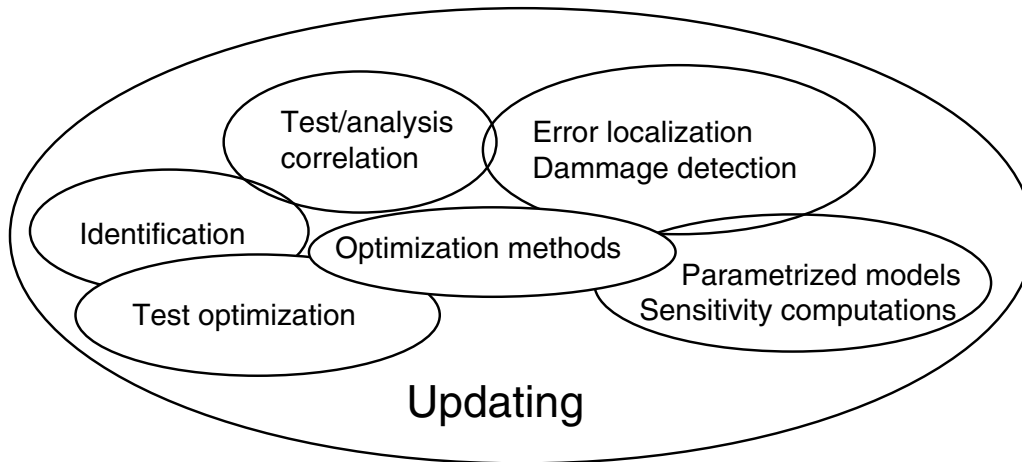


FIGURE 11.1 – Methods involved in updating

11.2 Model parametrization : localization, uncertainty,

...

Defining a family of possible models is the first step of any updating methodology. Updating is typically used in two main contexts

An experiment was specifically designed to validate a particular parameter known in advance. Parametrization is then straightforward and a priori steps are the verification of the visibility and discernability of the considered parameter. One thus runs a proper verification of the experiment will indeed measure the intended value.

A fairly generic experiment was performed and is used to validate the model. The parametrization now has the objective to detect incorrect parameters in the model. One thus runs a check of whether the answers found with updating are reliable or not. In this context, parametrization is typically composed of three steps

- engineering analysis to determine parts of the model that are likely to be incorrect,
- objective sensitivity to eliminate parameters that are not visible and thus cannot be determined,
- error localization techniques to try narrowing down on parameters that may be incorrect.

The use of generic experiments is also often considered in damage detection applications, where the range of possible damages can be large. The usefulness of the results then largely depends on the ability to narrow down on the possible damage scenarios.

11.2.1 Engineering selection

The objective of engineering selection is to determine parts that are likely to be incorrect.

The most common modeling errors are linked to the use of simplified meshes, in particular for joints. Ideally, one should verify through detailed local modeling that such simplified models are valid. This is done more and more often but not always and updating can then be used to detect a problem.

In most applications, one divides the structures in sets where a given physical parameter (thickness, modulus, density, ...) is assumed to be constant. In industrial applications, this division is often fixed once and for all (since in most FEM environments modifying this is difficult). One is thus far from the ideal procedure where a parameter corresponds to something that is effectively wrong.

[tiny Illustrations given in class should be shown here.](#)

11.2.2 Objective sensitivity

Most methods linked to the analysis model parameterization deal with the conditioning of the relation between physical parameters p and responses (modes, frequency responses, etc.). When running the optimization problem associated with updating, it is necessary that two sets of parameters lead to two different responses. Otherwise, the updating solution is not unique. This leads to the following questions

- is the parameter sensitive? If a parameter has no influence on the response then its value cannot be estimated by test/analysis correlation. This question motivates the need to physical parameters for sets of elements, since a single element rarely has a significant influence on the response. In statistical terms, one would say that the parameter values for a set of elements are perfectly correlated.
- has the parameter an influence that can be distinguished from that of other parameters. When the parameter/response relation is characterized by a linearized operator, this means that the operator is not practically degenerate.

One should be careful with a usual misunderstanding : insensitive parameters cannot be corrected reliably but still may be wrong. Sensitive parameters may be correct and are often modified to account for other errors in the test results or the model. In other words, one induces a bias in correct parameters to obtain a better correlation because the model cannot reproduce the test perfectly.

In practice, the sensitivity of all the test / analysis correlation criteria described in chapter 9 can be considered to validate the sensitivity of a given parameter. The most classical criterion is the frequency sensitivity which was shown to be directly proportional to the strain energy for stiffness variations and kinetic energy for mass variations.

For a set of parameters p_k , one thus computes the objective function sensitivity

$$\frac{dJ}{dp_k}(p) \quad (11.1)$$

One then classifies the effect of the various parameters and eliminates those whose influence is too small. Methods to compute objective sensitivities were described in chapter 10.

Figure 11.2 illustrates a typical display of MAC sensitivities for 15 correlated test modes of the Ariane 5 upper stage test. The sensitivity is represented by a square of varying size. Each column (paired mode) is normalized to the maximum sensitivity. The figure thus allows to detect parameters which never have a significant influence on the MAC correlation with any of the test modes (first two parameters *Maquette CU basse et CU haute* are among those).

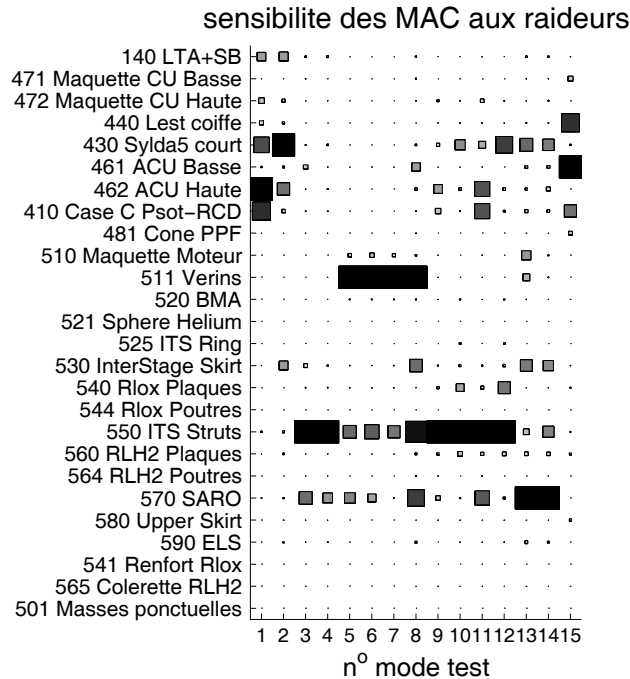


FIGURE 11.2 – For each pair of modes MAC sensitivity to change in stiffness parameter. Ariane 5 application [100].

Many such displays are possible and needed to be implemented for practical analysis of sensitivity studies.

Standard limitations in sensitivity analyses are the need to define a nominal variation range for each parameter, the correlation between parameters, the case of parameters whose nominal value is outside the sensitive range.

Sensitivity is defined for a parameter change of 1 but the definition of that unit is not obvious. For an uncertain parameter, one would use the standard deviation. In the absence of an uncertainty model, one typically defines the unit change as a fraction of the nominal value. Using a unit change of the value in its unit system can be very misleading since it depends on the unit system choice. The problem of parameter correlation can be illustrated as follows. The properties of one weld point have little influence on a car body, but if all weld points are modeled incorrectly in the same manner then the effect can be significant.

For parameters that can have very large variation ranges, such as equivalent springs often used to introduce equivalent joint models, the parameter can be insensitive around its nominal value. It is then important to check sensitivities around multiple operating points to ensure that the parameter is not inappropriately discarded.

11.2.3 Error localization

This section addresses error localization using energy methods (ERDC [127], MDRE [100], MECE [128]). These were first introduced in section 9.5.3.

For true modeshapes $\{\phi(p_0)\}$, the load residual is directly proportional to the variations in the dynamic stiffness $[Z(p) - Z(p_0)]$

$$R_{L_j}(\{\phi(p_0)\}, p) = [Z(p)] \{\phi(p_0)\} = [Z(p) - Z(p_0)] \{\phi(p_0)\} \quad (11.2)$$

For localization, one assumes that the expanded modeshapes solution of (9.37) are accurate representations of true modeshapes at all degree of freedom. The load residual resulting from (9.37) is thus assumed to a direct indications on the localization of non zero terms of $[Z(p) - Z(p_0)]$.

The direct use of the load residual for error localization induces many pathologies, since it is quite sensitive to mesh density, unit system choice, ... At a minimum, one must treat translations and rotations separately, but it preferable to use an energy measure and thus use the displacement residual (9.17).

When multiple displacement residuals j are available (multiple modes or test frequencies), one defines averages and can specify errors for each element or element group (e)

$$E^{(e)} = \frac{\sum_{j=1}^{N_{id}} \|R_j\|_{(e)}^2}{E_{Ref}} \quad (11.3)$$

Usual norms are strain and kinetic energy norms. While the choice of the norm has some influence, no choice seems obviously better and the strain energy is preferred in [100]. The reference error can be the norm of the displacement residual

$$E_{Ref} = \sum_{j=1}^{N_{id}} \|R_j\|^2 \quad (11.4)$$

thus allowing the definition of a relative error norm for the predicted quantity of interest.

Various measures of the relative error can be considered. The energy of modes of substructure i compared to global modal energy, with averaging of ratios or ratio of averages,

$$\begin{aligned} E_{1,\phi,i} &= \frac{1}{NM} \sum_{j=1}^{NM} \left(\frac{\sum_{el \in ss-struct.i} \phi_j^T K^{el} \phi_j}{\sum_{struct} \phi_j^T K^{el} \phi_j} \right), \\ E_{2,\phi,i} &= \frac{\frac{1}{NM} \sum_{j=1}^{NM} (\sum_{el \in ss-struct.i} \phi_j^T K^{el} \phi_j)}{\frac{1}{NM} \sum_{j=1}^{NM} (\sum_{struct} \phi_j^T K^{el} \phi_j)}, \end{aligned} \quad (11.5)$$

the mean energy of residuals relative to the mean energy of modes

$$\begin{aligned} E_{1,R_D,i} &= \frac{1}{NM} \sum_{j=1}^{NM} \left(\frac{\sum_{el \in ss-struct.i} R_{D,j}^T K^{el} R_{D,j}}{\sum_{struct} \phi_j^T K^{el} \phi_j} \right), \\ E_{2,R_D,i} &= \frac{\frac{1}{NM} \sum_{j=1}^{NM} (\sum_{el \in ss-struct.i} R_{D,j}^T K^{el} R_{D,j})}{\frac{1}{NM} \sum_{j=1}^{NM} (\sum_{struct} \phi_j^T K^{el} \phi_j)}, \end{aligned} \quad (11.6)$$

and the energy of residuals in substructure I relative to the global energy of the residuals

$$\begin{aligned} E'_{1,R_D,i} &= \frac{1}{NM} \sum_{j=1}^{NM} \left(\frac{\sum_{el \in ss-struct.i} R_{D,j}^T K^{el} R_{D,j}}{\sum_{el \in ss-struct.i} \phi_j^T K^{el} \phi_j} \right), \\ E'_{2,R_D,i} &= \frac{\frac{1}{NM} \sum_{j=1}^{NM} (\sum_{el \in ss-struct.i} R_{D,j}^T K^{el} R_{D,j})}{\frac{1}{NM} \sum_{j=1}^{NM} (\sum_{el \in ss-struct.i} \phi_j^T K^{el} \phi_j)}. \end{aligned} \quad (11.7)$$

11.3 Designing tests for localization

A key idea that has only been recently considered is the fact that the ability to localize should be checked *a priori*, thus giving better confidence in localization results.

11.3.1 Visibility and discernability

The first test seeks to verify that the localization criterion can indeed find error in a given substructure while indicating small errors in others. The idea, proposed in Ref. [128], is to determine for all possible residuals the one that localizes most of the error on a given substructure. If, for this residual, the energy of the residual is spread over many substructures then the localization is impossible.

For the MDRE, one can analyze the first row equation of (9.33) classically : the displacement residual is the static response to the load imbalance linked to the fact that the expanded modes do not verify equilibrium for a model that is incorrect. The second row equation however implies that

$$R_{Dj} = [Z_j]^{-1} [c]^T (\gamma Q_j (c\phi_{exp,j} - y_{Test,j})) \quad (11.8)$$

which means that the expansion limits residuals to linear combinations of responses to loads applied at sensors $[Z_j]^{-1} [c]^T$. The visibility of substructure i is thus linked to the maximization

$$V(i) = \max_u \frac{\{u\}^T [cZ_j^{-1} [K_i] Z_j^{-1} c^T] \{u\}}{\{u\}^T [cZ_j^{-1} [K] Z_j^{-1} c^T] \{u\}} \quad (11.9)$$

which is easily computed by solving the eigenvalue problem associated with this Raleigh quotient.

The visibility has values in the [0 1] interval and localization is theoretically perfect for a visibility of 1. Illustrations of this concept are given in Ref. [129].

In [100] visibility is discussed by assessing localization found when perturbing a given substructure and using pseudo tests corresponding to measurements of modes of this perturbed model. The conclusions found are similar : in many cases one cannot expect to localize errors. It is then necessary to redesign the sensor configuration with a particular localization objective.

By discernability studies, one seeks to establish the fact that an indication of error in a given substructure is dependent from errors in other substructures. Ref. [129] introduces criteria based on the notion of subspace angles. Ref [100] studies the impact on localization of residual errors on other parameters. Both studies give illustrations on specific examples that indicate a relatively poor capacity of energy criteria to distinguish errors.

The author's opinion is thus that error localization only works if the possible errors are known *a priori*, from engineering analysis of the problem, and that the ability to localize is verified *a priori* with the model.

11.3.2 Selective sensitivity

The need to have the objective function be sensitive to target parameters has been well illustrated. An original idea, known as the **selective sensitivity** method, seeks to reverse the

problem. One thus seeks to build an objective function that is sensitive to parameters in a set Ω_E and insensitive to all others

$$\left\| \frac{\partial R_j^{(e)}}{\partial p_k} (p^{(e)}) \right\|_Q = \begin{cases} \mathbf{high} & \text{if } p_k \in \Omega_E \\ \ll 1 & \text{if } (p_k) \notin C_{\Omega_E} \end{cases} \quad (11.10)$$

The method has been applied [130] using an output error criterion (9.40) and by adjusting input forces so as to optimize the verification of condition (11.10). Variants could be considered by adjusting the Q -norm applied to any correlation criterion expressed in the form of a residual. One then uses classical approaches to improve conditioning of an inverse problem by using non uniform weighting.

11.4 Global methods

11.4.1 Global (non parametric) vs. local (parametric)

One often groups methods that search perturbations $[\Delta M]$ and $[\Delta K]$ of initial matrices that minimize a given objective function under the name **global methods**. These perturbations are constrained by mathematical conditions on symmetry, positivity, sparsity, ... but not related to any physical interpretation. They can thus be called non-parametric.

This is by opposition to *local methods* (or parametric methods) that seek to describe the modifications as a set of changes to physical parameters describing the model (modulus, density, thickness, ...). The parametric vs. non parametric terminology is also used to describe uncertainties. Difficulties associated with parameterization were discussed in section 11.2.

For local methods (parametric models) the number of parameters can be more easily controlled and all correlation criteria introduced in chapter 9 can lead to an updating methodology by minimization of the associated objective functions.

In most cases, the optimization is posed as a form of non-linear least squares problem. Section 11.5.1 gives an overview of mathematical methods typically used to deal with this resolution. Sensitivity methods, the most common, are introduced in particular since many other approaches (simplex, genetic algorithms, ...) often require an excessive number of evaluations of the system which has a prohibitive cost even when using model reduction.

The number of parameters in non-parametric models is often quite large. Thus practical global algorithms are very limited in their optimization strategy. While thus use of global methods is not advocated here, understanding their basic working can be useful. A review of the rather abundant literature can be found in [131].

The existing literature almost always ignores the issue of spatial incompatibility between test and analysis. As a result methods are only applicable to small models where sensors can be used as DOFs. The author's understanding is that such cases are best dealt with using a modal model of the test and modeshape expansion for any relation with physical changes and this modal model. This has been done for SDM (Structural Dynamics Modification) applications in Corus [92] which is a class of hybrid test/analysis models.

If global update makes sense for hybrid model application, it is rarely useful for interpretation of physical parameters. Often one cannot relate matrix changes and physical parameter

errors unless there is a strong limitation on possible physical parameter changes. Such restrictions are the core contribution of local methods. Since there is no understanding of physical parameter changes involved, the interpretation of the validity of global updates is often dubious at best.

11.4.2 Common characteristics

The corrections to be performed on the mass and stiffness matrices are determined through algebraic equations without iteration whenever possible. The objectives and/or constraints are most often in the following list

- $\|\Delta M\|, \|\Delta K\|$ minimum norm change
- $[\Delta M]^T = [\Delta M], [\Delta K]^T = [\Delta K]$ symmetric matrices
- $[M + \Delta M] \geq 0, [K + \Delta K] \geq 0$ positive matrices
- $\text{profil}[M + \Delta M] = \text{profil}[M], \text{profil}[K + \Delta K] = \text{profil}[K]$ sparsity pattern typical of FE models
- $\|\Omega_{\text{id}}^2 - \Omega^2\|, \|\phi_{\text{id}} - \phi_1\|$ error on frequencies and modeshapes
- $\left\| \left(([K] + [\Delta K]) - \omega_{\text{id}_j}^2 ([M] + [\Delta M]) \right) \begin{Bmatrix} \phi_{\text{id}_j} \\ \phi_{2j} \end{Bmatrix} \right\|$ error on the modal residue
- Etc.

Since the unknown are the mass and stiffness changes, the Kuhn–Tucker conditions of optimality are written

$$\left[\frac{\partial J}{\partial \Delta M}(\Delta M; \Delta K) \right] = \mathbf{0}, \quad \left[\frac{\partial J}{\partial \Delta K}(\Delta M; \Delta K) \right] = \mathbf{0} \quad (11.11)$$

Furthermore, if the constraints are enforced using Lagrange multiplier, one needs to augment (11.11) with condition

$$\left\{ \frac{\partial J}{\partial \lambda}(\Delta M; \Delta K) \right\} = \mathbf{0} \quad (11.12)$$

Equations (11.11) and (11.12) are then manipulated to obtain an algebraic form of the optimal solution. Often the orthogonality conditions on modeshapes are also used to simplify computations

$$\{\phi_i\}^T [M] \{\phi_j\} = \mu_j \delta_{ij} \quad \{\phi_i\}^T [K] \{\phi_j\} = \mu_j \omega_j^2 \delta_{ij} \quad (11.13)$$

In the following sections, one considers that modes are mass normalized (that is $\mu_j = 1$).

11.4.3 Optimum matrix updating

Lorsque l'on s'intéresse à un recalage de la matrice de raideur par approche optimale, une des formulations proposées peut être de supposer la vérification de l'équation de définition des modes propres par les résultats du test

$$([K] + [\Delta K]) [\phi_{\text{id}}] = [M] [\phi_{\text{id}}] [\Omega_{\text{id}}]^2 \quad (11.14)$$

et la préservation de la symétrie de la matrice corrigée

$$([K] + [\Delta K])^T = ([K] + [\Delta K]) \quad (11.15)$$

On cherche alors, une correction de norme minimale permettant de vérifier les conditions posées. On peut par exemple utiliser

$$\min_{[\Delta K]} \left\| [M]^{-\frac{1}{2}} [\Delta K] [M]^{-\frac{1}{2}} \right\|_2^2 \quad (11.16)$$

où la pondération par rapport à la matrice de masse a pour but essentiel de simplifier les calculs, comme il sera vu dans la suite. Cette formulation suppose bien entendu que la matrice de masse est correcte (pas d'erreurs de modélisation en masse).

On peut calculer, à partir des conditions nécessaires de minimisation, la solution qui est obtenue sous la forme suivante :

$$\begin{aligned} [\Delta K] = & - [K] [W] [M] - [M] [W] [K] \\ & + [M] [W] [K] [W] [M] \\ & + [M] [\phi_{id}] [\Omega_{id}]^2 [\phi_{id}]^T [M] \end{aligned} \quad (11.17)$$

où

$$[W] = [\phi_{id}] [\phi_{id}]^T \quad (11.18)$$

L'implémentation de la solution optimale est donc très aisée puisqu'aucune résolution ni solveur itératif n'est nécessaire. Toutefois, cette approche n'est applicable que si tous les degrés de liberté sont instrumentés. De plus, rien ne garantit que la matrice corrigée soit définie positive ou qu'elle conserve sa structure creuse caractéristique de la connectivité des DDLs du modèle élément fini.

Plutôt que de formuler le problème en raideur, il est possible de rechercher la matrice $[\Delta K]$ qui va minimiser les changements apportés à la matrice de flexibilité statique, c'est à dire l'inverse de la raideur. La formulation peut alors s'écrire comme

$$\min_{[\Delta K]} \left\| [M]^{\frac{1}{2}} \left(([K] + [\Delta K])^{-1} - [K]^{-1} \right) [M]^{\frac{1}{2}} \right\|_2^2 \quad (11.19)$$

avec les mêmes deux contraintes que précédemment, c'est à dire la satisfaction de l'équation d'équilibre écrite à l'aide des paramètres modaux identifiés

$$([K] + [\Delta K]) [\phi_{id}] = [M] [\phi_{id}] [\Omega_{id}]^2 \quad (11.20)$$

et la symétrie de la matrice corrigée :

$$([K] + [\Delta K])^T = ([K] + [\Delta K]) \quad (11.21)$$

En fait, la solution obtenue par une formulation en flexibilité est encore plus simple que la précédente puisque l'on a

$$[\Delta K] = - [K] [\phi_{\text{id}}] [W]^{-1} [\phi_{\text{id}}]^T [K] + [M] [\phi_{\text{id}}] [\Omega_{\text{id}}]^2 [\phi_{\text{id}}]^T [M] \quad (11.22)$$

où la matrice $[W]$ représente la matrice de raideur projetée dans la base modale

$$[W] = [\phi_{\text{id}}]^T [K] [\phi_{\text{id}}] \quad (11.23)$$

L'inversion à calculer dans l'équation (11.22) n'est toutefois pas conséquente puisque la taille de $[W]$ est égale au nombre de modes identifiés (très exceptionnellement plus de 100). Cependant, rien ne garanti ici non plus le caractère défini positif de la matrice corrigée, ni la conservation de la structure creuse.

11.4.4 Correction globale à partir de FRF mesurées

Les raisons avancées pour justifier l'utilisation directe des FRFs pour le recalage sont évoquées en section 9.4. Pour le cas des méthodes globales, les itérations sont généralement basées sur la minimisation au sens des moindres carrés de

$$\min_{\Delta Z} \|([Z(s)] + [\Delta Z(s)]) \{y_{\text{id}}(s)\} - \{F(s)\}\| \quad (11.24)$$

où $\{y_{\text{id}}\}$ collecte les réponses en fréquence mesurées pour chaque paire d'entrées j et de sorties i et évaluées en un point de fréquence s particulier. De même, le vecteur $\{F\}$ représente les forces appliquées F_j évaluées à la fréquence s . Pour les tests habituels où les fonctions de transfert $H(s)$ sont mesurées, $F(s)$ devient une pondération (dans un premier temps on prendra 1 à toutes les fréquences) et $y(s)$ est donné par $H(s)F(s)$.

Comme solution de (11.24), on peut par exemple considérer

$$[\Delta Z(p; s)] = - [Z(p; s)] (\{y_{\text{id}}(s)\} - \{y(s)\}) \left(\{y_{\text{id}}(s)\}^T \{y_{\text{id}}(s)\} \right)^{-1} \{y_{\text{id}}(s)\}^T \quad (11.25)$$

ou utiliser une décomposition en valeurs singulières afin de reconstituer une matrice pseudo-inverse. Il faut remarquer que le terme à inverser entre parenthèses est scalaire et représente la norme Euclidienne du vecteur des fonctions de transfert mesurées. L'équation (11.25) conduit donc à une matrice de rang un, ce qui n'est de toute évidence pas acceptable physiquement. Afin d'obtenir une correction de la matrice dynamique de raideur qui soit de rang suffisant, il faut généraliser l'écriture (11.25) considérant les contributions en masse, amortissement et raideur, et en sommant sur suffisamment de points de fréquence

$$[\Delta Z(p; s)] = - \sum_{k=1 \dots N_s} \frac{[Z(p; s_k)] (\{y_{\text{id}}(s_k)\} - \{y(s_k)\}) \{y_{\text{id}}(s_k)\}^T}{\left(\{y_{\text{id}}(s_k)\}^T \{y_{\text{id}}(s_k)\} \right)} \quad (11.26)$$

Les inconvénients de cette approche peuvent être, comme dans le cas du recalage optimal des matrices : perte de la symétrie, perte du caractère défini-positif, perte du profil, difficulté d'interprétation du résultat.

De plus, la sélection des points de fréquence à utiliser dans l'équation (11.26) peut influencer fortement la reconstruction de la matrice $[\Delta Z]$.

11.5 Méthodes numériques pour le recalage

11.5.1 Moindres carrés non-linéaires et sensibilités

Le recalage est toujours posé comme un problème d'optimisation. Une fonction coût (généralement scalaire) est minimisée

$$\min_{\{p\}} J(p) \quad (11.27)$$

Les variables $\{p\}$ peuvent représenter des composantes des matrices éléments finis ou des paramètres de conception du modèle (modules, épaisseurs). Dans les deux cas, ce sont les variables d'optimisation qu'il s'agit de trouver comme solution du problème (11.27).

La condition nécessaire pour obtenir un extremum, dite condition de Kuhn–Tucker, s'écrit

$$\left\{ \frac{dJ}{dp}(p) \right\} = 0 \quad (11.28)$$

Cette équation constitue la base de toute implémentation numérique d'une méthode de recalage puisque c'est une condition nécessaire. Toutefois, il faudrait en toute rigueur vérifier également que l'extremum est un minimum local en montrant que le Hessien est localement défini positif

$$\{u\}^T \left[\frac{d^2J}{dp^2}(p) \right] \{u\} \geq 0, \quad \forall \{u\} \in \mathbb{R}^{NP} \quad (11.29)$$

En pratique, la vérification de la condition (11.29) est souvent difficile à obtenir : on se contente donc d'utiliser la condition nécessaire et de "trier" parmi les minima locaux vers lesquels l'algorithme pourrait converger. On se rappellera qu'une forme quadratique est positive si sa plus petite valeur propre est positive :

$$\lambda_{\min} \left(\left[\frac{d^2J}{dp^2}(p) \right] \right) \geq 0 \quad (11.30)$$

Une grande majorité des méthodes de recalage utilisent un coût représentant une norme quadratique d'un résidu $R(p)$ (que l'on précisera plus loin et qui est supposé réel ici pour simplifier les écritures). On a donc

$$J(p) = \|R\|_2^2 = \text{Trace} (R^T R) = \sum_{i,j} \bar{R}_{ij} R_{ij} \quad (11.31)$$

dont la dérivée première par rapport à p est donnée par

$$\frac{\partial J(p)}{\partial p} = 2\text{Trace} \left(\frac{\partial R^T}{\partial p} R \right) \quad (11.32)$$

et la dérivée seconde par

$$\frac{\partial^2 J(p)}{\partial p^2} = 2\text{Trace} \left(\frac{\partial R^T}{\partial p} \frac{\partial R}{\partial p} + R^T \frac{\partial^2 R}{\partial p^2} \right) \quad (11.33)$$

Une méthode d'optimisation de type Newton fait l'hypothèse qu'une approximation de la fonction par un développement limité d'ordre 2 est assez précise pour estimer la position de l'optimum. Étant donné la position actuelle p^n , l'estimation de la position suivante est $p^{n+1} = p^n + \delta p^{n+1}$ où δp^{n+1} est donné par

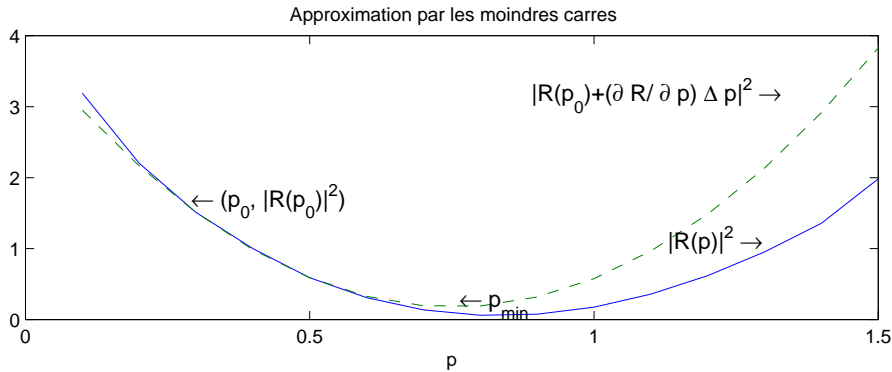
$$\left[\frac{\partial^2 J(p)}{\partial p^2} \right] \{ \delta p^{n+1} \} + \left\{ \frac{\partial J(p)}{\partial p} \right\} = \{ 0 \} \quad (11.34)$$

On se rappellera qu'il faudrait en théorie montrer que le **rayon spectral** de $\partial^2 J / \partial p^2$ (c'est à dire sa plus grande valeur propre) est inférieur à 1 pour avoir convergence

$$\lambda_{\max} \left(\left[\frac{d^2 J}{dp^2}(p) \right] \right) < 1 \quad (11.35)$$

mais on ne prend pas cette peine en pratique.

Les **méthodes de sensibilité** négligent le terme en $\partial^2 R / \partial p^2$ dans le calcul de la dérivée seconde du coût. Cette hypothèse permet est vérifiée si le résidu est une fonction linéaire des paramètres (ce qui arrive effectivement pour un certain nombre de méthodes de recalage). La simplification sur la forme du résidu, correspond à l'introduction d'un approximation quadratique de la fonction objectif dont on peut facilement calculer le minimum



L'itération considérée pour les méthodes de sensibilité est donc donnée par

$$\text{Trace} \left(\left\{ \frac{\partial R}{\partial p} \right\}^T \left(\frac{\partial R}{\partial p} \{ \delta p^{n+1} \} - R(p^n) \right) \right) = 0 \quad (11.36)$$

Il apparaît clairement dans cette écriture que chaque itération correspond à la minimisation

$$\min_{\{ \delta p^{n+1} \}} \left\| \left\{ \frac{\partial R}{\partial p} \right\} \{ \delta p^{n+1} \} - \{ R(p^n) \} \right\| \quad (11.37)$$

par annulation de la projection sur les sensibilités des résidus.

La résolution de problèmes de la forme (11.37) est généralement connue sous la forme de résolution de problèmes de moindres carrés linéaires (la notation habituelle est la minimisation de la norme quadratique de $Ax - b$). Il n'est pas certain que la solution du problème de moindre carré donnée par (11.36) soit la plus efficace et on se souviendra donc à bon escient des méthodes

mathématiques développées pour la résolution de problèmes de moindres carrés (voir section A).

Notez enfin, que même si à chaque itération, on résout des problèmes de moindres carrés linéaires, le processus global de recalage reste cependant non-linéaire car le résidu et les sensibilités dépendent de p .

11.5.2 Recalage et optimisation sous contrainte

On rajoute souvent pour le recalage un certain nombre de **contraintes**. Celles-ci permettent de prendre en compte des conditions d'orthogonalité, de veiller à ce que l'équation d'équilibre reste satisfaite tout au long de la minimisation, etc. Elles permettent aussi d'écartier a priori un certain nombre de solutions qui pourraient s'avérer non-physiques.

La prise en compte des contraintes dans la minimisation

$$\min_{\{p\}} J(p) \quad \text{avec} \quad \mathcal{C}(p) = 0 \quad (11.38)$$

peut se faire soit par pénalité soit par l'utilisation de multiplicateurs de Lagrange (pour ne citer que les deux types d'approches les plus courantes).

Les méthodes de **pénalisation** imposent la vérification des contraintes en ajoutant à la fonction coût un terme supplémentaire qui devient très grand quand les contraintes ne sont pas satisfaites

$$\tilde{J}(p) = J(p) + \{\sigma\}^T \{\mathcal{C}(p)\} \quad (11.39)$$

Les coefficients $\{\sigma\}$ représentent des poids constants fixés par l'utilisateur. (Ceux-ci peuvent éventuellement évoluer au cours de la minimisation). Ce sont des constantes qui ne constituent donc pas des variables d'optimisation supplémentaires.

L'approche des **multiplicateurs de Lagrange**, contrairement à la pénalisation, ajoute des variables supplémentaires $\{\lambda\}$ et utilise le coût augmenté (appelé Lagrangien)

$$\mathcal{L}(p; \lambda) = J(p) + \{\lambda\}^T \{\mathcal{C}(p)\} \quad (11.40)$$

Le Lagrangien est stationnaire par rapport à p et λ au point cherché. La condition nécessaire de Kuhn–Tucker devient donc

$$\left\{ \frac{\partial \mathcal{L}}{\partial p}(p; \lambda) \right\} = 0 \quad \text{et} \quad \left\{ \frac{\partial \mathcal{L}}{\partial \lambda}(p; \lambda) \right\} = 0 \quad (11.41)$$

puisque la première dérivée totale de la fonction à minimiser \mathcal{L} doit être nulle. En utilisant la définition (11.40) du Lagrangien, il vient

$$\left\{ \frac{dJ}{dp}(p) \right\} + \left[\frac{d\mathcal{C}}{dp}(p) \right]^T \{\lambda\} = \mathbf{0} \quad \text{et} \quad \{\mathcal{C}(p)\} = \mathbf{0} \quad (11.42)$$

Une interprétation mécanique des multiplicateurs de Lagrange est de les considérer comme des “forces de rappel” s'exerçant sur le système afin de remettre à zéro les contraintes. L'introduction de multiplicateurs de Lagrange conduit à des problèmes de point-selle : on cherche à minimiser la fonction \mathcal{L} selon les variables $\{p\}$ et à la maximiser selon les variables $\{\lambda\}$.

La méthode de Lagrange de base est souvent modifiée. On peut par exemple utiliser un Lagrangien augmenté en incorporant une partie pénalisée afin d'en accélérer la convergence

$$\mathcal{L}(p; \lambda) = J(p) + \{\lambda\}^T \{\mathcal{C}(p)\} + \frac{1}{2} \{\mathcal{C}(p)\}^T [\sigma] \{\mathcal{C}(p)\} \quad (11.43)$$

D'un point de vue pratique, les équation (11.42) sont souvent non-linéaires et couplés. On peut donc utiliser un algorithme de Newton comme pour la minimisation sans contraintes. Une itération est définie par

$$\begin{bmatrix} \frac{\partial^2 J(p)}{\partial p^2} & \frac{\partial^2 J(p)}{\partial p \partial \lambda} \\ \frac{\partial^2 J(p)}{\partial \lambda \partial p} & \frac{\partial^2 J(p)}{\partial \lambda^2} \end{bmatrix} \begin{Bmatrix} \delta p^{n+1} \\ \delta \lambda^{n+1} \end{Bmatrix} + \begin{Bmatrix} \frac{\partial J(p)}{\partial p} \\ \frac{\partial J(p)}{\partial \lambda} \end{Bmatrix} = \{0\} \quad (11.44)$$

qui compte tenu de l'expression du Lagrangien correspond à

$$\begin{bmatrix} \frac{d^2 J}{dp^2}(p^{(n)}) & \frac{d\mathcal{C}}{dp}{}^T(p^{(n)}) \\ \frac{d\mathcal{C}}{dp}(p^{(n)}) & 0 \end{bmatrix} \begin{Bmatrix} \delta p^{(n+1)} \\ \lambda^{(n+1)} \end{Bmatrix} = - \begin{Bmatrix} \frac{dJ}{dp}(p^{(n)}) \\ \mathcal{C}(p^{(n)}) \end{Bmatrix} \quad (11.45)$$

où les paramètres d'optimisation sont réévalués comme $\{p^{(n+1)}\} = \{p^{(n)}\} + \{\delta p^{(n+1)}\}$. Il convient de remarquer que

1. Le bloc zéro dans la matrice à inverser (11.45) peut générer des difficultés lors de la factorisation (pivots nuls) mais cette difficulté disparaît avec l'utilisation d'un Lagrangien augmenté.
2. Les contraintes et la fonction coût ne sont pas traitées de manière symétrique. Cette dernière est dérivée deux fois tandis que les contraintes ne sont dérivées qu'une seule fois. Il convient donc de payer attention au choix des contraintes et fonctions coûts.

11.5.3 Moindres carrés et statistique

On considère un problème de moindres carrés

$$[A] \{\delta p\} - (\{b\} + \{e\}) = \{0\} \quad (11.46)$$

dans lequel on suppose que les quantités $\{b\}$, dérivant d'une mesure expérimentale ou d'un modèle inexact, sont connues avec une erreur $\{e\}$ de **moyenne nulle**. On suppose, et **c'est la grande difficulté**, pouvoir estimer la matrice de covariance des erreurs e

$$\text{Var}[e] = [W] \quad (11.47)$$

En supposant que la distribution des paramètres e est gaussienne on peut montrer que la solution de l'équation (11.46) au sens des moindres carrés généralisés représentant le meilleur estimateur linéaire non-biaisé (ou estimateur "blue," Best Linear Unbiased Estimator) de $\{\delta p\}$ est donné par

$$\{\delta p\} = - \left([A]^T [W]^{-1} [A] \right)^{-1} [A]^T [W]^{-1} \{g\} \quad (11.48)$$

De plus, l'erreur moyenne associée à la solution "blue" est donnée par

$$\xi^2 = (\{g\} + [H] \{\delta p\})^T [W]^{-1} (\{g\} + [H] \{\delta p\}) \quad (11.49)$$

De cette erreur moyenne peut être déduite la matrice de variance/covariance de la solution (11.48)

$$\text{Var} [\delta p] = \xi^2 \left([A]^T [W]^{-1} [A] \right) \quad (11.50)$$

On voit donc que l'approche statistique permet non seulement de trouver la solution au problème de moindres carrés, mais aussi de proposer une estimation de la confiance qu'il est possible d'accorder à cette solution. Cette confiance est directement mesurée par les coefficients de la matrice de variance/covariance : intuitivement, le k -ème paramètre p_k dans $\{\delta p\}$ est corrigé avec un bon niveau de confiance si la variance associée $(\text{Var} [\delta p])_{kk}$ est petite.

En pratique, les biais sur les quantités expérimentales sont souvent supérieurs à la variance liée au bruit de mesure et les covariances estimées ne sont que des valeurs arbitraires représentant la confiance perçue par l'utilisateur dans les diverses quantités estimées. Les méthodes statistiques sont donc utilisées en dehors du cadre qui a conduit à leur formulation.

Cette limitation n'enlève en fait pas grand chose de leur intérêt qui est de pré-conditionner le problème de moindres carrés avec une information a priori sur la confiance accordée aux diverses quantités expérimentales en définissant la variance (11.47) et d'analyser la sensibilité du résultat à des perturbations réalistes en étudiant la variance du résultat (11.50). Dans le cas du recalage, une incertitude peut être utilisée pour

- prendre en compte les erreurs de mesure et d'identification
- prendre en compte les incertitudes de modélisation
- estimer la robustesse du recalage
- estimer la qualité du modèle avec les variances des paramètres de conception

Les coûts, souvent importants, des calculs pour la mise à jour des matrices de variance/covariance ainsi que les difficultés d'implémentation qui limitent souvent l'application de ces techniques à des situations simples où l'utilisation d'une covariance n'est pas différente de l'utilisation d'un préconditionneur.

11.5.4 Méthodes de régularisation

La régularisation prend sa source avant la technique de filtrage via SVD (ou autre) exposée ci-dessus. Elle consiste à résoudre un problème similaire mais très légèrement modifié afin d'essayer d'en améliorer le conditionnement.

Régulariser un problème consiste en effet à résoudre la minimisation suivante

$$\min_{\{\delta p\}} J(p^n) + \sigma \{p^n - p^0\}^T \{p^n - p^0\} \quad (11.51)$$

où un terme a été rajouté afin de pénaliser les solutions p^n qui apporteraient trop de changement par rapport au modèle de départ. En plus de ses effets numériques bénéfiques, la pénalisation possède donc aussi un sens physique précis : écarter les solutions qui préconisent un changement trop important du modèle de départ.

La modification de la fonction objectif conduit à des itérations de la forme

$$\left[\frac{\partial^2 J(p)}{\partial p^2} + \sigma [I] \right] \{ \delta p^{n+1} \} + \left\{ \frac{\partial J(p)}{\partial p} + \sigma \{ p^n - p^0 \} \right\} = \{ 0 \} \quad (11.52)$$

que l'on peut écrire sous la forme

$$[H_\sigma^{(n)}] \{ \delta p_\sigma^{(n+1)} \} = - \{ g_\sigma^{(n)} \} \quad (11.53)$$

avec

$$\{ g_\sigma^n \} = \left\{ \frac{dJ}{dp}(p^n) \right\} + \sigma \{ p^n - p^0 \} \quad [H_\sigma^n] = \left[\frac{d^2 J}{dp^2}(p^n) \right] + \sigma [I] \quad (11.54)$$

On voit qu'en rajoutant $\sigma [I]$ à la matrice H^n on améliore son conditionnement puisque des valeurs singulières très inférieures à σ sont remplacées par σ alors que celles beaucoup plus grandes ne sont pas modifiées.

11.6 Conclusion

Finite element model updating has two main uses. It is used to determine a few parameters known in advance through experiments designed specifically for that purpose. It is used to help engineers validate global models of complex structures by giving an experimental reference of the *true* response.

Typical limitations of the experimental reference are

- signal processing (excitation and measurement noise and bias, aliasing, leakage, ...)
- experimental topology (poor positioning/orientation of sensors)
- instrumentation (poor calibration, incorrect measurement of force, ...)
- identification bias (poor pole estimates, poor identification of residual terms, bias induced by non-linearity, FRF estimation, ...)
- the structure tested is one of a class, there should be an uncertainty model associated with the test result.

On the model side, typical sources of problems are

- poor mesh convergence in certain areas or poor element selection. Such problems can be avoided by proper checks, they should be eliminated in a verification phase.
- simplified representations of complex assemblies or junctions by simple *equivalent* models (springs for bolted connection, ...). The validity of equivalent models can generally be established using local studies with fine 3D models. The overall cost of such detailed studies is typically small compared to the cost of a test but there is always a cost trade-off.
- damping is not accounted for or grossly simplified.
- non linearities (contact, friction, non linear materials, ...) are ignored.

Updating itself, has its own problems

- correlation criterion selection

- spatial incompatibility
- modal truncation
- implementation issues for optimization of large models
- interfacing of FEM and optimization software

While a few years back, updating literature discussed correlation criteria a lot. It now seems that when the procedure works well, all criteria give similar results. Thus remaining issues are more related to

- how to account for test problems and give engineers practical tools to obtain statistical quantifications of test variability and possibly limit that variability.
- how to rapidly test multiple updating strategies and verify their coherence. Updating is then seen as a range of methods to test the validity of models of complex structures rather than actually determine model parameters.
- how to implement the optimization phase of updating efficiently for large order models.
- how to validate mesh convergence for applications in dynamics
- how to exploit tests to build parameter uncertainty models and how to use these models in robust design procedures.

Chapitre 12

Damping basics

12.1 Basic damping mechanisms

Damping is a broad term encompassing the descriptions of various behavior and mechanisms.

A reasonable definition would be as follows. A system is damped if the amount of energy it contains decreases due transformation to a form of energy that is not considered to be part of of the system.

Common examples would be

- the damper of a car typically contains a cavity with a small hole through which a fluid moves at high speeds. Viscosity of the fluid leads to heat elevation. From the mechanical point of view, energy is dissipated.
- viscoelasticity xxx
- plasticity within a material or friction in an interface xxx
- shock damping : [132].
- thermoelastic damping

12.1.1 Linear viscoelasticity/complex modulus

Viscoelastic damping for a material is well described in reference books [133, 134, 135]. Linear viscoelastic materials (also called Boltzmannian materials) assume a linear relation between the history of loading and the history of response.

For stress and strain for example, this translates into the existence of a relaxation function $h(t)$ given by

$$\sigma(t) = \int_0^{\infty} \varepsilon(t - \tau)h(\tau)d\tau \quad (12.1)$$

Using Laplace transform, one sees that this hypothesis is equivalent to the existence of a complex modulus $C(s)$ (transform of $h(t)$) such that

$$\{\sigma(s)\} = [C(s)] \{\varepsilon(s)\} = [C'(s) + iC''(s)] \{\varepsilon(s)\} \quad (12.2)$$

Mechanical engineers typically use a reologic representation of viscoelastic behavior as a Maxwell model.

12.1.2 Friction/plasticity/hysteretic materials

A general definition of hysteretic behavior is the dependence of the state of a system on the system history and not only the current states (Larousse, Wikipedia, ...).

When applied to a mechanical constitutive law, hysteresis is the fact that stress will depend on the history of strain. This definition obviously encompasses viscoelasticity but due to the frequency transfer the stress depends on the rate of the strain history (xxx if going more slowly the xxx). A number of systems however are such that the dependence on state history is independent of rate. This is in particular the case of friction in surfaces or plasticity in materials.

Coulomb friction is a model where the tangential force on presumed rigid is proportional to the normal force when sliding occurs

12.1.3 Coupling with an external system

12.2 Damping representation at a system level

When performing a system level analysis detailed dissipation models may be averaged in both space and time.

- Material versus/system.

- Modal damping assumption.

- System level hysteretic damping.

Annexe A

Appendix

A.1 English/French Glossary

pairing	appariement
updating	recalage
state (vector describing a model, for mechanical applications state vectors combine displacement and velocity)	état (vecteur décrivant un modèle, pour les applications mécanique le vecteur d'état combine déplacement et vitesse)
state space model	modèle d'état
DOF (degree of freedom)	DDL (degré de liberté)
SVD singular value decomposition	DVS décomposition en valeurs singulières
loss factor	facteur de perte
constitutive law	loi de comportement
static extension (lift)	relèvement statique
wire frame	modèle filaire

A.2 Décomposition en valeurs singulières

Pour toute matrice rectangulaire $H_{n \times m}$ à valeurs réelles ou complexes, on définit

- σ_j les racines carrées strictement positives des valeurs propres non nulles de $H^H H$ (ou de manière équivalente de HH^H) appelées **valeurs singulières**

$$\sigma_j(H) = \sqrt{\lambda_j(H^H H)} = \sqrt{\lambda_j(HH^H)} \quad (\text{A.1})$$

- U_j les vecteurs propres droits de HH^H appelés **vecteurs singuliers à gauche**. Par définition, ils vérifient $[HH^H] \{U_j\} = \sigma_j^2 \{U_j\}$ et, par hypothèse, ils sont normalisés à l'unité ($\{U_j\}^H \{U_j\} = 1$). En utilisant la propriété d'orthogonalité des modes propres d'un opérateur auto-adjoint, on montre que la matrice $[U]_{n \times n} = [U_1 \ \dots \ U_n]$ des vecteurs singuliers gauche est unitaire

$$\{U_j\}^H \{U_k\} = \delta_{jk} \quad (\text{A.2})$$

- V_j les vecteurs propres droits de $H^H H$ appelés **vecteurs singuliers à droite**. Par définition, ils vérifient $[H^H H] \{V_j\} = \sigma_j^2 \{V_j\}$ et, par hypothèse, ils sont normalisés à l'unité ($\{V_j\}^H \{V_j\} = 1$). En utilisant la propriété d'orthogonalité des modes propres d'un opérateur auto-adjoint, on montre que la matrice $[V]_{n \times n} = [\{V_1\} \ \dots \ \{V_m\}]$ des vecteurs singuliers gauche est unitaire

$$\{V_j\}^H \{V_k\} = \delta_{jk} \quad (\text{A.3})$$

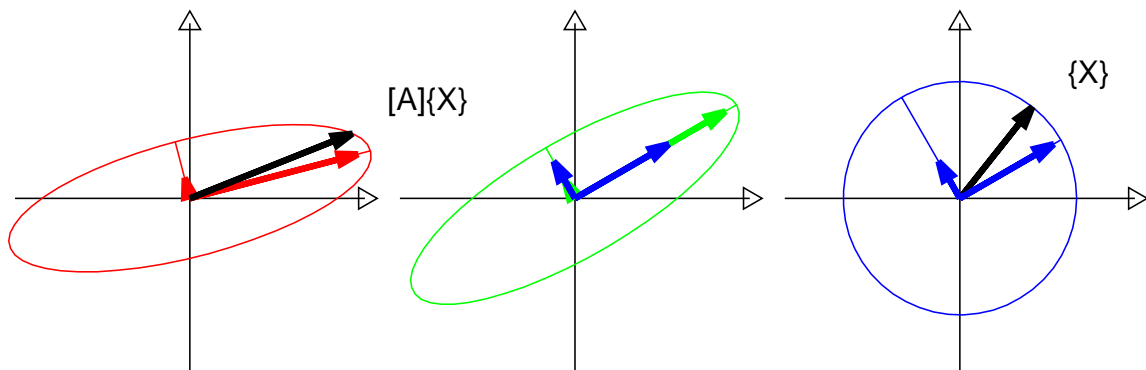
- Σ matrice diagonale des valeurs singulières

$$[\Sigma] = \left[\begin{array}{cccc|c} \sigma_1 & 0 & \dots & 0 & [0]_{k \times (m-k)} \\ 0 & \sigma_2 & \dots & 0 & \\ \dots & \dots & \dots & \dots & \\ 0 & \dots & 0 & \sigma_k & \\ \hline [0]_{(n-k) \times k} & & & & [0]_{(n-k) \times (m-k)} \end{array} \right] \quad (\text{A.4})$$

est donnée par $[\Sigma] = [U]^H [H] [V]$ et conduit à la **décomposition en valeurs singulières** (SVD pour "singular value decomposition" en anglais) de H sous la forme

$$[H] = [U] [\Sigma] [V]^H \quad (\text{A.5})$$

L'interprétation géométrique de la décomposition en valeurs singulières d'une matrice réelle est simple. En lisant le dessin ci-dessous de droite à gauche comme le produit matriciel, une entrée u de norme 1 (donc sur le cercle unitaire) est projetée sur la base orthogonale des vecteurs singuliers à gauche V_k . Chaque composante est amplifiée par un facteur correspondant aux valeurs singulières σ_k (la sphère est transformée en un ellipsoïde). On effectue enfin une rotation de l'ellipsoïde pour passer des axes principaux V_k aux axes principaux U_k correspondant aux vecteurs singuliers droits.



Vous avez déjà utilisé la notion de décomposition en valeurs singulières en parlant de moments d'inertie principaux ou de contraintes principales. Dans ces applications la matrice

décomposée est cependant symétrique alors que la SVD est définie pour des matrices arbitraires (rectangulaires, réelles ou complexes, etc.). On peut montrer que les méthodes de sous-structuration dynamique procèdent aussi de l'utilisation d'une décomposition en valeurs singulières avec la norme en masse sur l'espace d'entrée et la norme en énergie de déformation sur l'espace de sortie [52].

Vous pourrez utiliser à bon escient les relations additionnelles sur les valeurs singulières données par

$$\begin{aligned}
\sigma_{\max}(A^{-1}) &= \frac{1}{\sigma_{\min}(A)} \\
\sigma_{\min}(A^{-1}) &= \frac{1}{\sigma_{\max}(A)} \\
\sigma_{\max}(A) - 1 &\leq \sigma_{\max}(I + A) \leq \sigma_{\max}(A) + 1 \\
\sigma_{\min}(A) - 1 &\leq \sigma_{\min}(I + A) \leq \sigma_{\min}(A) + 1 \\
\sigma_{\max}(A + B) &\leq \sigma_{\max}(A) + \sigma_{\max}(B) \\
\sigma_{\max}(AB) &\leq \sigma_{\max}(A)\sigma_{\max}(B)
\end{aligned} \tag{A.6}$$

Pour les applications en identification, la décomposition en valeurs singulières est utilisée

- pour déterminer la dimension réelle du sous espace engendré par une très grande matrice.
- pour obtenir une solution bien conditionnée d'un problème de moindres carrés (voir section suivante).

A.3 Moindres carrés et conditionnement numérique

Les problèmes d'identification sont souvent exprimés en termes de problèmes de moindres carrés linéaires de la forme générale

$$\min_{\{x\}} \|[A] \{x\} - [b]\|_2^2 \tag{A.7}$$

La résolution de problème de moindres carrés est un problème bien traité mathématiquement (voir par exemple [136]) mais souvent mal compris en pratique. On insiste ici, sur les problèmes de conditionnement numérique.

L'identification faisant intervenir des quantités expérimentales, on est amené à supposer que A et/ou b sont perturbés par de "petites" erreurs. La solution du système (A.7) est donc elle aussi perturbée par une quantité $\{\delta x\}$ qui vérifie

$$([A] + [\delta A]) (\{x\} + \{\delta x\}) = (\{b\} + \{\delta b\}) \tag{A.8}$$

La solution du problème n'est bien définie que si pour de "petites" perturbations, la solution perturbée ne diffère que peu de la solution initiale, soit

$$\frac{\|\delta A\|}{\|A\|} \ll 1, \quad \frac{\|\delta b\|}{\|b\|} \ll 1 \quad \implies \quad \frac{\|\delta x\|}{\|x\|} \ll 1 \tag{A.9}$$

La mesure du conditionnement de la matrice A permet de savoir dans quelle mesure l'implication (A.9) est vérifiée. La mesure du conditionnement demande de passer par la norme

d'opérateur de A définie de la manière suivante. On peut montrer qu'il existe un nombre $c = \|\|A\|\|$, appelé **norme matricielle**, dépendant uniquement de A tel que

$$\|[A] \{x\}\| \leq c \|\{x\}\| \quad (\text{A.10})$$

On démontre dans tout bon livre sur l'algèbre linéaire que

$$\frac{\|\delta x\|}{\|x\|} \leq \kappa(A) \left(\frac{\|\|\delta A\|\|}{\|\|A\|\|} + \frac{\|\delta b\|}{\|b\|} \right) \quad (\text{A.11})$$

où $\kappa(A)$ appelé **nombre de conditionnement** est défini par

$$\kappa(A) = \|\|A\|\| \|\|A^{-1}\|\| \quad (\text{A.12})$$

Cette condition permet de donner une borne supérieure sur les perturbations susceptibles d'être apportées à la solution d'un système linéaire lorsque celui-ci est modifié. On remarque le rôle important du conditionnement puisqu'une grande valeur $\kappa(A)$ va pouvoir amplifier toute perturbation introduite, aussi petite soit-elle.

L'implication (A.9) est donc satisfaite quand **la matrice $[A]$ est bien conditionnée**. Dans le cas d'une matrice **mal conditionnée**, la perturbation $\{\delta x\}$ peut devenir très importante comparée à la solution initiale $\{x\}$ même si les changements apportés au système sont négligeables. Physiquement, ce mauvais conditionnement numérique est la traduction du fait que la solution recherchée n'est pas unique, d'où un problème d'indétermination.

Attention : le mauvais conditionnement numérique peut avoir de nombreuses sources. Le problème peut être effectivement mal posé (il existe effectivement plusieurs solutions minimisant le critère), les unités de différents variables peuvent être mal choisies (exemple : DDL de rotation/translation), etc.

La norme matricielle de A se base sur les choix de norme pour l'espace des $\{x\}$ et l'espace des $[A] \{x\}$. La mesure du conditionnement est donc intimement lié aux normes choisies. On pourra utilement se rappeler que pour comparer des champs mécaniques les normes en énergie cinétique et énergie de déformations sont des alternatives souvent plus "physiques" que la norme euclidienne qui donnera par exemple la même pondération à des DDL de translation et de rotation.

Lorsqu'un problème est mal conditionné, la première chose à faire est d'essayer, dans la mesure du possible, de changer sa formulation afin de lever les indéterminations qui sont la source du mauvais conditionnement. Si cela n'est pas possible, on peut soit tenter de filtrer les composantes responsables de la propagation du mauvais conditionnement (l'outil approprié est la décomposition en valeurs singulières comme détaillé ci-dessous), soit essayer de régulariser le problème avant de le résoudre.

A.4 Moindres carrés et SVD

Une manière de résoudre, un problème de moindres carrés efficacement est d'utiliser la décomposition en valeurs singulières de A

$$[A] = \sum_{j=1}^{\min(n,m)} \sigma_j \{U_j\} \{V_j\}^H \quad (\text{A.13})$$

Étant donné une décomposition en valeurs singulières de la matrice A , on montre facilement que

$$\{x\} = \sum_{j=1}^{\min(n,m)} \sigma_j^{-1} \{V_j\} \{U_j\}^H [b] = [V] \left[\backslash \sigma_j^{-1} \backslash \right] [U] [b] \quad (\text{A.14})$$

est la solution du problème de moindres carrés s'il n'en existe qu'une seule. On note $[A]^+$ la matrice $[V] \left[\backslash \sigma_j^{-1} \backslash \right] [U]$ et on peut vérifier qu'il s'agit un **pseudo inverse au sens de Moore-Penrose** c'est à dire une matrice vérifiant

$$[A^+] [A] [A^+] = [A^+] \quad (\text{A.15})$$

$$[A] [A^+] [A] = [A] \quad (\text{A.16})$$

$$([A] [A^+])^T = ([A] [A^+]) \quad (\text{A.17})$$

$$([A^+] [A])^T = ([A^+] [A]) \quad (\text{A.18})$$

La solution (A.14) n'existe clairement que si toutes les valeurs singulières sont non nulles. Or la définition du zéro expérimental/numérique est quelque chose de difficile. On est donc amené à reposer le problème en termes de conditionnement numérique.

On peut assez facilement démontrer que le nombre de conditionnement est donné par

$$\kappa(A) = \frac{\sigma_{\max}([A])}{\sigma_{\min}([A])} \quad (\text{A.19})$$

soit le rapport de la plus grande valeur singulière de la matrice par la plus petite. De toute évidence, une matrice singulière (dont la plus petite valeur singulière est alors égale à zéro) peut, d'après cette définition, posséder un nombre de conditionnement qui tend vers l'infini. On voit aussi qu'une matrice est bien conditionnée si toutes ses valeurs propres possèdent des ordres de grandeur similaires de façon à ce que le rapport (A.19) ne devienne pas trop important.

La nombre de conditionnement donne une indication de la possibilité numérique de trouver une solution unique à un problème de moindres carrés (pour A carrée la possibilité d'inverser A). La SVD de la matrice A peut s'écrire sous la forme

$$[A] = \sum_{j=1}^{\min n,m} \sigma_j \{U_j\} \{V_j\}^H \quad (\text{A.20})$$

Les vecteurs U_j et V_j étant de norme 1 les ordres de grandeur des produits $\{U_j\} \{V_j\}^T$ sont tous identiques. En calcul flottant double précision l'addition de deux nombre n'est significative que si leurs ordres de grandeur diffèrent de moins de 10^{16} . Une matrice dont le nombre de conditionnement dépasse cette valeur n'est donc à priori pas inversible dans une représentation en flottant double précision. En pratique, elle peut l'être quand même mais cela n'a rien de

systematique. Les problèmes de conditionnement numérique sont TRÈS courants en calcul de structure.

Dans les cas de l'identification et du recalage, il est utile d'estimer le niveau de précision sur la connaissance de la matrice A et de le traduire en termes de niveau minimal de réponse permettant d'obtenir une information significative (pouvoir distinguer du bruit la contribution d'une valeur singulière donnée). En pratique, ceci se traduit par la sélection d'un nombre NR de valeurs singulières significatives

$$[\Sigma] = \begin{bmatrix} \sigma_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \sigma_{NR} & & & & & \\ & & & \epsilon & & & & \\ & & & & \ddots & & & \\ & & & & & & \epsilon & \\ & & & & & & & \epsilon \end{bmatrix} \quad (\text{A.21})$$

les autres (notées ϵ) étant supposées ne pas apporter d'information physique (être égales à zéro). On construit alors un pseudo-inverse dont la solution est située dans le sous-espace engendré par les NR premiers vecteurs singuliers à gauche de la décomposition en valeurs singulières de A

$$\{x\} = [A]^+ [b] = \left[\sum_{j=1}^{NR} \sigma_j^{-1} \{V_j\} \{U_j\}^T \right] [b] \quad (\text{A.22})$$

Ce pseudo-inverse évite donc de propager des informations liées à de petites valeurs singulières car elles sont susceptibles d'être fortement modifiées par une petite modification de A . En d'autres termes, on ne cherche une solution que dans un sous-espace de dimension NR , ce qui permet d'améliorer le conditionnement du problème posé. Le sous espace sélectionné correspond à celui donnant un conditionnement optimal.

En conclusion n'oubliez pas que le conditionnement dépend des normes utilisées pour x et b . Il change donc si on

- change la norme utilisée dans l'espace des b . On parle de préconditionnement et cela correspond à multiplier $Ax - b$ à gauche par une matrice donnée. Les méthodes d'identification paramétrique statistique (voir section suivante) et de sensibilités sélectives rentrent dans cette catégorie.
- change la norme utilisée sur l'espace des x , il s'agit encore de préconditionnement. On pourra par exemple définir comme unitaire le changement de chacun des paramètres considérés comme raisonnable.
- élimine certains x (sélection des paramètres à recalage)

A.5 Basic continuous mechanics equations

More details can be found in many books, Ref.[137] for example. The equations for traction/compression in local and integral (principle of virtual works, form)

	local	PVW
On Ω	$\rho A \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} (EA \frac{\partial u}{\partial x}) = f_v(x)$	$\int_{\Omega} \rho A \hat{u} \ddot{u} + EA \hat{u}_{,x} u_{,x} = F_{ext}$
Right fixed	$u(x_R) = 0$	$u(x_R) = 0$
Right free	$F_x = EA u_{,x} _{x_R} = 0$	-
Spring at right	$EA u_{,x}(x) - K u(x) = 0$	$\int_{\Omega} \dots + K \hat{u}(x) u(x) = F_{ext}$

The force applied to the right of the beam to achieve equilibrium is given by $F_x = EA u_{,x}$
 Check that you know how to deal with local dashpot, local mass, periodic solution

The equations for beam bending in local and integral (principle of virtual works, form)

	local	PVW
On Ω	$\rho A \frac{\partial^2 w}{\partial t^2} + EI \partial u^4 \partial x^4 = f_v(x)$	$\int_{\Omega} \rho A \hat{w} \ddot{w} + EI \hat{w}_{,x^2} w_{,x^2} = F_{ext}$
Right fixed	$w(x_R) = 0, w_{,x} = 0$	$w(x_R) = 0, w_{,x} = 0$
Right pinned	$w(x_R) = 0, EI w_{,x^2} = 0$	$w(x_R) = 0$
Right free	$M_y = EI w_{,x^2} = 0, F_z = -EI w_{,x^3} = 0$	-
Spring	$EI w_{,x^2} = 0, EI w_{,x^3}(x) - K w(x) = 0$	$\int_{\Omega} \dots + K \hat{w}(x) w(x) = F_{ext}$

Check that you know how to deal with local torsion spring, mass, dashpot.

The last classical continuous model is that of plate bending

$$D \Delta \Delta w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \tag{A.23}$$

with $D = \frac{Eh^3}{12(1-\nu^2)}$.

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