A.4 Beam problem

A.4.1 Text

We consider a beam which length is $4L$, cantilevered at both ends and submitted to a bending force F in his middle (see figure 21). Because of the symmetry of the problem, we only study one half of the beam by priscribing a zero rotation condition in the middle. $v(x)$ denotes the vertical displacement of the points of the beam and the Euler-Bernoulli approximation is considered. A

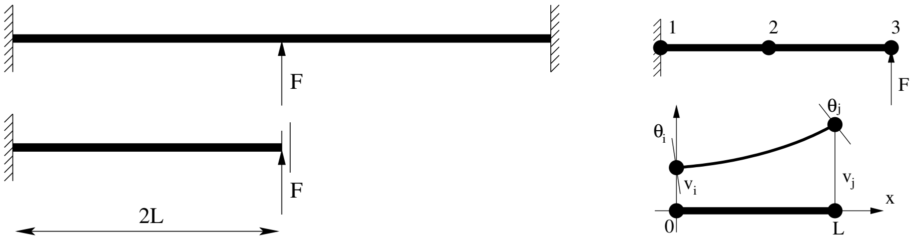

Figure 21: Beam, symmetry and discretisation

finite element discretisation is used on this half-beam. It contains two elements with a L length (figure 21 right). v_i denotes the value of the displacement field $v(x)$ on node i. θ_i denotes the section rotation on node i. Using the Euler-Bernoulli approximation, the displacement field $v(x)$ on such an element is:

$$v(x) = \phi_i(x)v_i + \varphi_i(x)\theta_i + \phi_j(x)v_j + \varphi_j(x)\theta_j$$

with

$$\phi_i(x) = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}, \quad \varphi_i(x) = x - \frac{2x^2}{L} + \frac{x^3}{L^2}, \quad \phi_j(x) = \frac{3x^2}{L^2} - \frac{2x^3}{L^3}, \quad \varphi_j(x) = \frac{x^2}{L} + \frac{x^3}{L^2}$$

Using this discretisation, as the element all have the same length L, the elementary stiffness matrix on one element is:

$$[K_{el}] = a \begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix} \text{ avec } a = \frac{2EI}{L^3} \quad ; \quad \text{associated dof: } \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix}$$

where E is the Young's modulus of the material, I the bending inertia of the section and L the length of an element.

Part 1 - Boundary conditions

1. Assemble the global stiffness matrix and the generalized force vector.
2. Give the matrix form of the boundary condition.
3. Built the system that have to be solved when those conditions are taken into account using the *substitution* technique.
4. Solve the problem and draw the shape of the beam.
5. show that, when the system is condensed on the displacement degrees of freedom v_2 and v_3, it becomes:

$$\begin{bmatrix} 12a & -6a \\ -6a & \frac{15a}{4} \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix} \quad \text{where } v_1 = \theta_1 = \theta_3 = 0 \quad \text{and} \quad \theta_2 = \frac{3v_3}{4L}$$

This last system is the one used in the following.

Part 2 : frictionless unilateral contact with gap The displacement of the beam is now limitated by the presence of a rigid body (figure 22). j is The distance between the membrane and the rigid body.

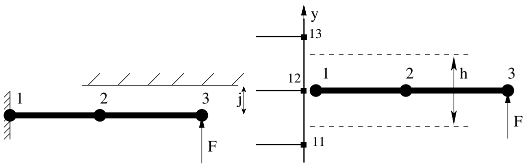

Figure 22: Frictionless contact with gap and model connection

1. Write the local contact conditions between the beam and the rigi body.
2. Give the matrix form of the discrete displacement condition. Explain why this not equivalence between the discrete form and the continuous form of this condition.
3. Solve the problem using the status method in the case where $j = \frac{F}{2a}$. For that, on shall take the conditions into account using the Lagrange multiplier method in which λ_i will denote the multiplier associated to displacement v_i. Draw the deformed shape of the beam.

   Note : we shall recall that the Lagrange multiplier is the opposite of the reaction force on the contact zone.
4. Is the displacement condition satisfied at each point of element $2-3$?
5. We now want to prescribed the non penetration condition as a mean condition on element $2-3$. Write the condition on the displacement and rotation degrees of freedom corresponding to a constant force mean connection.

Incompatible model connection The left cantilevered side is now replaced by a connection with deformable media (figure 22 right). $u_m(x)$ and $v_m(x)$ denotes the horizontal and vetical components of the deformable media. A linear finite element discretisation of the media is used: the node located on the connection are indicated on the figure. They have a uniform size e. In the connection region, with such a discretisation, the considered fields are locally expressed:

$$u_m(x) = \begin{cases} \dfrac{u_{13}-u_{12}}{e} y + u_{12} \quad , & \text{if } y \in [0,e] \\ \dfrac{u_{12}-u_{11}}{e} y + u_{12} \quad , & \text{if } y \in [-e,0] \end{cases} ; v_m(x) = \begin{cases} \dfrac{v_{13}-v_{12}}{e} y + v_{12} \quad , & \text{if } y \in [0,e] \\ \dfrac{v_{12}-v_{11}}{e} y + v_{12} \quad , & \text{if } y \in [-e,0] \end{cases}$$

1. Write the ponctual connection conditions between the degrees of freedom of the media and the ones of the node 1 of the beam when the rotation of the media is expressed from the displacement of nodes 11 and 13.
2. Write the mean connection conditions that are to be prescribed on the degrees of freedom for the transmission of forces and moment knowing that the real thickness of the beam is h.

A.4.2 Correction

Part 1 - Boundary conditions

1. Stiffness matrix and generalized force vector. The system is:

$$a\begin{bmatrix} 6 & 3L & -6 & 3L & 0 & 0 \\ 3L & 2L^2 & -3L & L^2 & 0 & 0 \\ -6 & -3L & 12 & 0 & -6 & 3L \\ 3L & L^2 & 0 & 4L^2 & -3L & L^2 \\ 0 & 0 & -6 & -3L & 6 & -3L \\ 0 & 0 & 3L & L^2 & -3L & 2L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ F \\ 0 \end{Bmatrix}$$

2. Matrix form of the conditions

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

3. system to be solved:

$$\begin{bmatrix} 12 & 0 & -6 \\ 0 & 4L^2 & -3L \\ -6 & -3L & 6 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F \end{Bmatrix}$$

4. Solution:

$$v_2 = \frac{2F}{3a}; \qquad \theta_2 = \frac{F}{aL}; \qquad v_3 = \frac{4F}{3a}$$

5. Condensing the system on dof v_2 and v_3, we obtain:

$$\begin{bmatrix} 12a & -6a \\ -6a & \frac{15a}{4} \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix} \quad \text{and a post-computation gives} \quad \theta_2 = \frac{3v_3}{4L}$$

Part 2 - frictionless unilateral contact with gap

1. Contact conditions: there is no friction and the interaction is only in the vertical direction:

$$v(x) \leq j \quad ; \quad f(x) \leq 0 \quad ; \quad f(x).(v(x) - j) = 0$$

where $f(x)$ is the force density of the body on the membrane.

2. Discrete form of the displacement condition:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \end{Bmatrix} \leq \begin{Bmatrix} j \\ j \end{Bmatrix}$$

3. Resolution using the status method:

   (a) The strict condition is prescribed:

$$\begin{bmatrix} 12a & -6a & 1 & 0 \\ -6a & \frac{15a}{4} & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \\ \lambda_2 \\ \lambda_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \\ j \\ j \end{Bmatrix}$$

which solution is:

$$\left\{\begin{array}{llll}\lambda_2 & = -6aj & = -3F & < 0 \\ \lambda_3 & = F + \frac{9}{4}aj & = \frac{17F}{8} & > 0\end{array}\right. \quad ; \quad \left\{\begin{array}{lll} v_2 & = j & = \frac{F}{2a} \\ v_3 & = j & = \frac{F}{2a}\end{array}\right.$$

Reaction forces on nodes 2 is positive so the condition on this node have to be removed.

(b) Strict condition on node 3 is only prescribed:

$$\begin{bmatrix} 12a & -6a & 0 \\ -6a & \frac{15a}{4} & 1 \\ 0 & 1 & 0 \end{bmatrix} \left\{\begin{array}{c} v_2 \\ v_3 \\ \lambda_3 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ F \\ j \end{array}\right\}$$

the solution is then:

$$\lambda_3 = \frac{5}{8}F > 0 \quad ; \quad \left\{\begin{array}{llll} v_2 & = \dfrac{j}{2} & = \dfrac{F}{4a} & \leq j \\ v_3 & = j & = \dfrac{F}{2a} & \leq j \end{array}\right.$$

Reaction force on node 3 is negative, the displacement conditions are satisfied.

(c) The condition is satisfied on each point

(d) Mean condition on element $2-3$:

$$\int_0^L (v(x) - j)dx \leq 0$$

that gives after intgration, using the basis function and the fact that $\theta_3 = 0$:

$$\frac{L}{2}v_2 - \frac{L^2}{4}\theta_2 + \frac{L}{2}v_3 - Lj \leq 0$$

so the condition to be prescribed is:

$$\frac{1}{2}v_2 - \frac{L}{4}\theta_2 + \frac{1}{2}v_3 - j \leq 0$$

Part 3 - Contact with incompatible meshes

1. The ponctual conditions are:

$$u_1 = u_{12} \quad ; \quad v_1 = v_{12} \quad ,; \quad \theta_1 = \frac{u_{11} - u_{13}}{2e}$$

2. The mean conditions that allow the transmission of forces and moment are:

$$\int_{-h/2}^{h/2} \underline{F}_i^* (\underline{u}_m(y) - \underline{u}_p(y))dy = 0, \forall i = 1, 2, 3$$

where:

$$\underline{F}_1^* = \underline{x} \quad ; \quad \underline{F}_2^* = \underline{y} \quad ; \quad \underline{F}_3^* = y\underline{x}$$

what gives:

$$\left\{\begin{array}{l} \displaystyle\int_{-h/2}^{h/2} (u_m(y) - u_p(y))dy = 0 \\ \displaystyle\int_{-h/2}^{h/2} (v_m(y) - v_p(y))dy = 0 \\ \displaystyle\int_{-h/2}^{h/2} y(u_m(y) - u_p(y))dy = 0 \end{array}\right.$$

with, for the beam displacement:

$$u_p(y) = u_1 - \theta_1 y \quad ; \quad v_p(y) = v_1$$

what gives:

$$\begin{cases} \displaystyle\int_{-h/2}^{h/2} u_m(y)dy - hu_1 & = 0 \\ \displaystyle\int_{-h/2}^{h/2} v_m(y)dy - hv_1 & = 0 \\ \displaystyle\int_{-h/2}^{h/2} yu_m(y)dy + \frac{h^3}{12}\theta_1 & = 0 \end{cases}$$

Using the expression of the displacement of the media, one obtains:

$$\begin{cases} u_1 & = \dfrac{h}{8e}(u_{11} + u_{13} - 2u_{12}) + 2u_{12} \\ v_1 & = \dfrac{h}{8e}(v_{11} + v_{13} - 2v_{12}) + 2v_{12} \\ \theta_1 & = \dfrac{u_{11} - u_{13}}{2e} \end{cases}$$

which are the conditions that are to be prescribed.