A.3 Membrane problem

A.3.1 Text

We use a 1d approximation of a membrane submited to a tension T and to an extrenal pressure p on its lower surface. The two extremities are clamped in order that the vertical displacement is nul (figure 15). $v(x)$ denotes the vertical displacement of the nodes of the membrane.

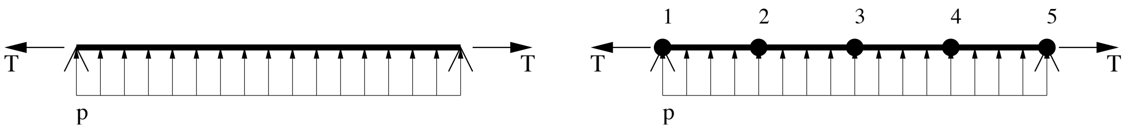

Figure 15: Membrane under pressure and discretisation

Part 1 - Boundary conditions A finite element discretisation of the membrane is used. It presents four linear element which length is e (figure 15 right).

v_i denotes the value of the field $v(x)$ on node i. As the ellemnt all have the same length, the elementary stiffness matrix and the elementary generalized force vector, corresponding to a distributed pressure p, are:

$$[K_{el}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \quad ; \quad \{F_{el}\} = \left\{ \begin{array}{c} \frac{pe}{2} \\ \frac{pe}{2} \end{array} \right\}$$

where $k = \frac{T}{e}$ is the membrane stiffness. In the following, the value k is kept in the expression of the stiffness matrices.

1. Assemble the global stiffness matrix and the generalized force vector.
2. Give the matrix form of the boundary condition.
3. Built the system that have to be solved when those conditions are taken into account using the *substitution* technique.
4. Solve the problem and draw the shape of the membrane.

In the following, we'll keep in assembled matrix in which we have eliminate the prescribed degrees of freedom.

Part 2 : frictionless unilateral contact with gap The displacement of the membrane is now limitated by the presence of a rigid body (figure 16). j is The distance between the membrane and the rigid body.

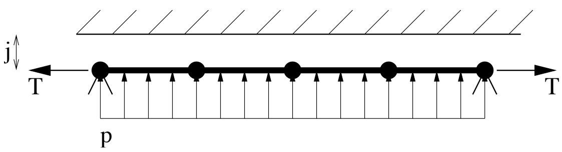

Figure 16: Frictionless contact with gap

1. Write the local contact conditions between the membrane and the rigi body.
2. Give the matrix form of the discrete displacement condition. Is there an equivalence between the discrete condition and the continuous one?

3. Solve the problem using the status method in the case where $j = \frac{3pe}{2k}$. For that, on shall take the conditions into account using the Lagrange multiplier method in which λ_i will denote the multiplier associated to node i. Draw the deformed shape.

   Note : we shall recall that the Lagrange multiplier is the oppsite of the reaction force on the contact zone.

Contact with incompatible meshes The membrane can now come into contact with a deformable body whose width is e situated at a distance j (figure 17). $v_m(x)$ denotes the vertical

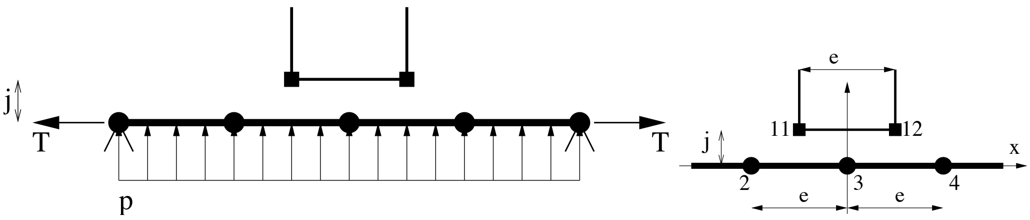

Figure 17: Incompatible meshes

displacement of the membrane and $v_s(x)$ the one of the body.

We use a linear finite elemnt discretisation of the vertical displacement of the solid wich is not compatible with the one of the membrane (wich is the same than the one used in the other parts). Figure 17 (right) gives the local parametrization.

Regarding the discretisations, the displacement fields can be expressed, on the contact zone, by:

$$v_s(x) = \frac{v_{12} - v_{11}}{e} x + \frac{v_{12} + v_{11}}{2}, \quad \forall x \in [-\frac{e}{2}, \frac{e}{2}] \quad ; \quad v_m(x) = \begin{cases} \frac{v_4 - v_3}{e} x + v_3 & , \quad \text{if } x \in [0, e] \\ \frac{v_3 - v_2}{e} x + v_3 & , \quad \text{if } x \in [-e, 0] \end{cases}$$

1. Write the continuous local frictionless contact conditions with gap between the deformable body and the membrane.
2. explain the notions of master and slave meshes for taking into account the nodale non penetration conditions.
3. Explain the notion of *connection at a mean sens*.
4. Give the conditions that have to be taken into account on the degrees of freedom depending on the type of connection:
   (a) ponctual connection for which mesh of the deformable solid is the master one.
   (b) ponctual connection for which mesh of the membrane is the master one.
   (c) connection at a mean sens allowing the transmission of constant forces.
   (d) connection at a mean sens allowing the transmission of linear forces.
5. We now consider that the body is rigid and fixed.
   (a) Give the discrete condition on the membraner when the conditions are taken into account using the technique proposed at 4(c).
   (b) Solve the problem using the status method.
   (c) Is the non penetration condition satisfied every where on the contact zone?

A.3.2 Correction

Part 1 - Boundary conditions

1. Stiffness matrix and generalized force vector:

$$[K] = \begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix} \quad ; \quad \{F\} = \begin{Bmatrix} \frac{pe}{2} \\ pe \\ pe \\ pe \\ \frac{pe}{2} \end{Bmatrix}$$

2. Matrix form of the conditions

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

3. system to be solved:

$$\begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} pe \\ pe \\ pe \end{Bmatrix}$$

4. Solution:

$$v_2 = \frac{3pe}{2k}; \qquad v_3 = \frac{2pe}{k}; \qquad v_4 = \frac{3pe}{2k}$$

Part 2 - frictionless unilateral contact with gap

1. Contact conditions: there is no friction and the interaction is inly in the vertical direction:

$$v(x) \leq j \quad ; \quad f(x) \leq 0 \quad ; \quad f(x).v(x) = 0$$

   where $f(x)$ is the force density of the body on the membrane.

2. Discrete form of the displacement condition:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \end{Bmatrix} \leq \begin{Bmatrix} j \\ j \\ j \end{Bmatrix}$$

3. Resolution using the status method:

   (a) The strict condition is prescribed:

$$\begin{bmatrix} 2k & -k & 0 & 1 & 0 & 0 \\ -k & 2k & -k & 0 & 1 & 0 \\ 0 & -k & 2k & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{Bmatrix} = \begin{Bmatrix} pe \\ pe \\ pe \\ j \\ j \\ j \end{Bmatrix}$$

   which solution is:

$$\begin{cases} \lambda_2 & = pe - kj = -\frac{pe}{2} & < 0 \\ \lambda_3 & = pe & > 0 \\ \lambda_4 & = pe - kj = -\frac{pe}{2} & < 0 \end{cases} \quad ; \quad \begin{cases} v_2 & = j & = \frac{3pe}{2} \\ v_3 & = j & = \frac{3pe}{2} \\ v_4 & = j & = \frac{3pe}{2} \end{cases}$$

Reaction forces on nodes 2 and 4 are positive so the conditions on these nodes have to be removed.

(b) Strict condition on node 3 is only prescribed:

$$\begin{bmatrix} 2k & -k & 0 & 0 \\ -k & 2k & -k & 1 \\ 0 & -k & 2k & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \\ \lambda_3 \end{Bmatrix} = \begin{Bmatrix} pe \\ pe \\ pe \\ j \end{Bmatrix}$$

the solution is then:

$$\lambda_2 = pe > 0 \quad ; \quad \begin{cases} v_2 & = \dfrac{pe}{2k} + kj & = \dfrac{5pe}{4k} \\ v_3 & = j & = \dfrac{3pe}{2k} \\ v_4 & = \dfrac{pe}{2k} + kj & = \dfrac{5pe}{4k} \end{cases}$$

Reaction force on node 3 is negative, the displacement conditions are satisfied.

Part3 - Contact with incompatible meshes

1. Frictionless contact conditions:

$$u_m(x) - u_s(x) \leq j$$

2. Master and slave meshes (see lecture notes)
3. Connection at a mean sens (see lecture notes)
4. Discrete condition depending on the type of connection:

(a) ponctual connection for which mesh of the deformable solid is the master one:

$$u_3 - \frac{u_{12} + u_{11}}{2} \leq j$$

(b) ponctual connection for which mesh of the membrane is the master one:

$$\begin{cases} \dfrac{u_2 + u_3}{2} - u_{11} \leq j \\ \dfrac{u_4 + u_3}{2} - u_{12} \leq j \end{cases}$$

(c) connection at a mean sens allowing the transmission of constant forces:

$$\int_{-e/2}^{e/2} (u_m(x) - u_s(x) - j) dx \leq 0$$

thus, after integration:

$$\frac{1}{8} v_2 + \frac{1}{8} v_4 + \frac{3}{4} v_3 - \frac{1}{2} v_{12} - \frac{1}{8} v_{11} \leq j$$

(d) connection at a mean sens allowing the transmission of linear forces. The following condition have to be added to the previous one:

$$\int_{-e/2}^{e/2} x(u_m(x) - u_s(x) - j) dx \leq 0$$

that is, after integration:

$$v_4 - v_2 - 2v_{12} + 2v_{11} \leq 0$$

5. We now consider that the body is rigid and fixed.

   (a) The contact condition becomes:

$$\frac{1}{8}v_2 + \frac{1}{8}v_4 + \frac{3}{4}v_3 \leq j$$

   (b) Resolution by the status method:

$$\begin{bmatrix} 2k & -k & 0 & \frac{1}{8} \\ -k & 2k & -k & \frac{3}{4} \\ 0 & -k & 2k & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} pe \\ pe \\ pe \\ j \end{Bmatrix}$$

   what gives:

$$\lambda = \frac{5}{2}pe - \frac{4}{3}kj$$

   using the value of j proposed in section 2, one have:

$$\lambda = \frac{1}{2}pe > 0$$

   The reaction force is negative, the condition is then satisfied. The solution is then:

$$\lambda = \frac{1}{2}pe \qquad ; \qquad \begin{cases} v_2 &= \dfrac{5}{4}\dfrac{pe}{k} \\ v_3 &= \dfrac{25}{16}\dfrac{pe}{k} \\ v_2 &= \dfrac{5}{4}\dfrac{pe}{k} \end{cases}$$

   (c) One have $v_3 > j$ thus the non penetration condition is not satisfied locally on node 3.

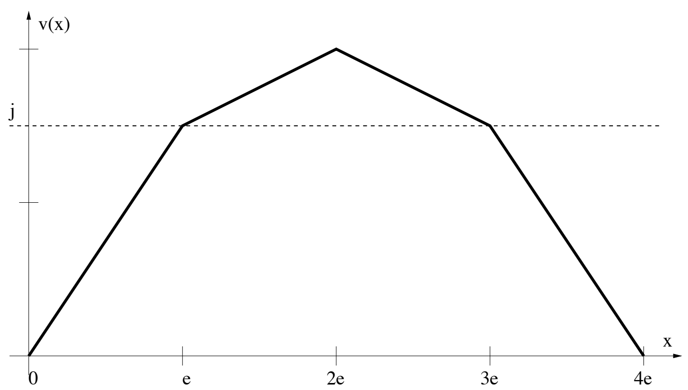

Figure 18: Solution of part 1

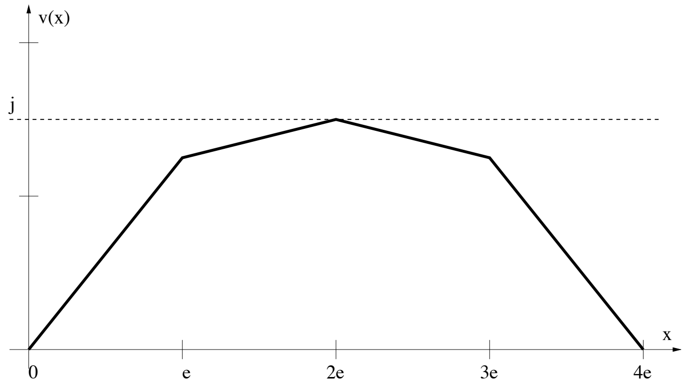

Figure 19: Solution of part 2

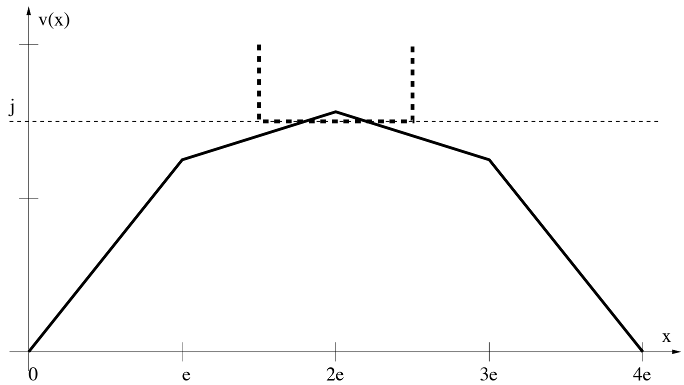

Figure 20: Solution of part 3