A Exercices

A.1 Boundary conditions on a lattice

A.1.1 Text

We consider the three bar lattice presented on figure 13 (left).

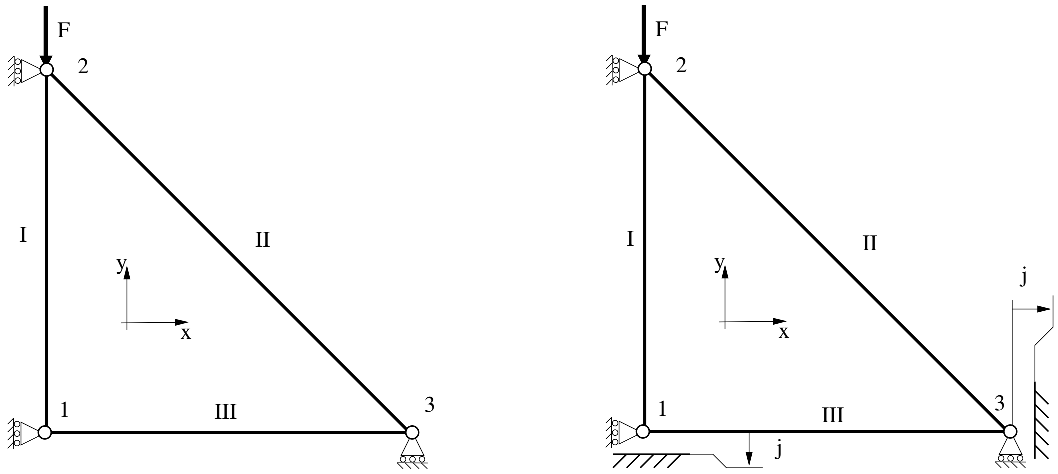

Figure 13: Lattice : isostatical (left) and with unilateral supports (right)

u_{x_i} and u_{y_i} denote the values of displacement of node i. The geometrical and material caracteristics of the bars are such that the stiffness matrix of the bars are:

$$[K_I] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k & 0 & -k \\ 0 & 0 & 0 & 0 \\ 0 & -k & 0 & k \end{bmatrix} \begin{array}{c} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \end{array}, [K_{II}] = \begin{bmatrix} k & -k & -k & k \\ -k & k & k & -k \\ -k & k & k & -k \\ k & -k & -k & k \end{bmatrix}, \begin{array}{c} u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \end{array},$$

$$[K_{III}] = \begin{bmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} u_{x_1} \\ u_{y_1} \\ u_{x_3} \\ u_{y_3} \end{array}$$

1. Assemble the stiffness matrix of the lattice as well as the vector of forces.
2. Write de boundary conditions presented on the left figure in a matrix form.
3. Build the given system when those condition are taken into account by *substitution*.
4. Solve the problem for the case presented on the left figure.
5. We now also lock the degree of freedom u_{x_3}. Solve by taking into account this new condition with the lagrange multiplier method applied two the system built at question 3. Give the reaction force in direction x on the support of node 3.
6. We now add unilateral frictionless supports with gaps on node 1 and 3, such as presented on the right figure. The gap is the same on the two support and denoted j. It is such that $j = \frac{3F}{2k}$.

   (a) Write de unilateral conditions on the degrees of freedom and on the reaction forces.

(b) Solve the system by using the *status method*. In this approach, the conditions on the dof will be prescribed using the lagrange multiplier method. Gibe the displecement of the nodes and the reaction force in the supports for the solution.

Note: we shall recall that the lagrange multiplier is the opposite of the reaction force.

A.1.2 Correction

1. After assembly, the system to be solved is:

$$\begin{bmatrix} k & 0 & 0 & 0 & -k & 0 \\ 0 & k & 0 & -k & 0 & 0 \\ 0 & 0 & k & -k & -k & k \\ 0 & -k & -k & 2k & k & -k \\ -k & 0 & -k & k & 2k & -k \\ 0 & 0 & k & -k & -k & k \end{bmatrix} \begin{Bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -F \\ 0 \\ 0 \end{Bmatrix}$$

2. The boundary conditions can be written in a matrix form:

$$\left\{ \begin{array}{l} u_{x_1} = 0 \\ u_{x_2} = 0 \\ u_{y_3} = 0 \end{array} \right. \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

3. When those conditions are taken into account using the substitution technique, there are only three unknowns left and to system to be solved is:

$$\begin{bmatrix} k & -k & 0 \\ -k & 2k & k \\ 0 & k & 2k \end{bmatrix} \begin{Bmatrix} u_{y_1} \\ u_{y_2} \\ u_{x_3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -F \\ 0 \end{Bmatrix}$$

4. The resolution of the systel gives:

$$u_{y_1} = \frac{F}{k} \quad ; \quad u_{y_2} = -\frac{2F}{k} \quad ; \quad u_{x_3} = -\frac{2F}{k}$$

5. When adding the condition $u_{x_3} = 0$, using the lagrange multiplier method, the system becomes:

$$\begin{bmatrix} k & -k & 0 & 0 \\ -k & 2k & k & 0 \\ 0 & k & 2k & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u_{y_1} \\ u_{y_2} \\ u_{x_3} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} 0 \\ -F \\ 0 \\ 0 \end{Bmatrix}$$

which solution is:

$$u_{y_1} = \frac{F}{k} \quad ; \quad u_{y_2} = -\frac{F}{k} \quad ; \quad u_{x_3} = 0 \quad ; \quad \lambda = F$$

Reaction on node 3 in direction x is then: $F_{x_3} = -\lambda = -F$.

6. We add unilateral frictionless supports on nodes 1 and 3 with gap j such that$j = \frac{3F}{2k}$.

   (a) The unilateral conditions are:

$$\left\{ \begin{array}{l} u_{y_1} \geq -j \\ u_{x_3} \leq j \end{array} \right. \quad \text{et} \quad \left\{ \begin{array}{l} F_{y_1} \geq 0 \\ F_{x_3} \leq 0 \end{array} \right.$$

(b) Solution by the status method:

Stage 1 Solve using the strict conditions:

$$\left\{ \begin{array}{l} u_{y_1} = -j \\ u_{x_3} = j \end{array} \right. \Rightarrow \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right] \left\{ \begin{array}{c} u_{y_1} \\ u_{y_2} \\ u_{x_3} \end{array} \right\} = \left\{ \begin{array}{r} -j \\ j \end{array} \right\}$$

using the Lagrange multiplier technique. The system is:

$$\left[\begin{array}{rrrrr} k & -k & 0 & 1 & 0 \\ -k & 2k & k & 0 & 0 \\ 0 & k & 2k & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right] \left\{ \begin{array}{c} u_{y_1} \\ u_{y_2} \\ u_{x_3} \\ \lambda_1 \\ \lambda_3 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ -F \\ 0 \\ -j \\ j \end{array} \right\}$$

which solution is:

$$u_{y_1} = -j = -\frac{3F}{2k} \quad ; \quad u_{y_2} = -\frac{2F}{k} \quad ; \quad u_{x_3} = j = \frac{3F}{2k} \quad ; \quad \lambda_1 = -\frac{F}{2} \quad ; \quad \lambda_3 = -F$$

We check the conditions on the forces:

$$\left\{ \begin{array}{ll} F_{y_1} = -\lambda_1 \geq 0 & \text{(satisfied condition, will stay)} \\ F_{x_3} = -\lambda_3 \geq 0 & \text{(non satisfied condition, should be suppressed)} \end{array} \right.$$

Stage 2 Solve the system prescribing the condition that where keeped at the end of last stage:

$$\left\{ \; u_{y_1} = -j \right. \Rightarrow \left[\begin{array}{ccc} 1 & 0 & 0 \end{array} \right] \left\{ \begin{array}{c} u_{y_1} \\ u_{y_2} \\ u_{x_3} \end{array} \right\} = \left\{ \; -j \; \right\}$$

using the Lagrange multiplier technique. The system is:

$$\left[\begin{array}{rrrr} k & -k & 0 & 1 \\ -k & 2k & k & 0 \\ 0 & k & 2k & 0 \\ 1 & 0 & 0 & 0 \end{array} \right] \left\{ \begin{array}{c} u_{y_1} \\ u_{y_2} \\ u_{x_3} \\ \lambda_1 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ -F \\ 0 \\ -j \end{array} \right\}$$

which solution is:

$$u_{y_1} = -j = -\frac{3F}{2k} \quad ; \quad u_{y_2} = -\frac{5F}{3k} \quad ; \quad u_{x_3} = \frac{5F}{6k} \quad ; \quad \lambda_1 = -\frac{F}{6}$$

We check the conditions on the forces:

$$\left\{ \begin{array}{ll} F_{y_1} = -\lambda_1 = \frac{F}{6} \geq 0 & \text{OK} \\ u_{y_1} = -j & \text{OK} \\ u_{x_3} = -\frac{F}{6} \leq j & \text{OK} \end{array} \right.$$

The calculated solution is then the right one.