Simulations numériques pour la dynamique
Réduction de modèle

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Ensam/PIMM, SDTools

A few activities

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d) F. Conejos, e) R. Penas, f) H. Pinault
Why does SDTools exist?

- **Necessity:** programmatic access to all steps
- **Chosen solution:** flexible toolbox & custom applications
  - Experimental Modal Analysis
  - Test / Analysis correlation
  - 3D Finite Element Modeling
- **With a modular approach**
  - MATLAB environment
  - OpenFEM: Core software for Finite Element Modeling (co-developed with INRIA)
  - FEMlink: import / export industrial modules
  - Runtime SDT: customized and standalone compiled applications

CAD/Meshing
FEM
Simulation
Testing

CATIA, Workbench, …
NASTRAN, ABAQUS, ANSYS,…
Adams, Simpack, Simulink,…
Siemens TestLab, ME-Scope,…
What is a system?

- **Inputs** $u(t)$: hammer with force measurement
- **Outputs** $y(t)$
  - Test: vibrometer on testbed
  - Computation: stresses
- **State** $x(t)$
  - Displacement & velocity field as function of time
    \[
    \{\dot{x}(t)\} = f(x(t), u(t), p, t) \quad \text{evolution}
    \]
    \[
    \{y(t)\} = g(x(t), u(t), p, t) \quad \text{observation}
    \]
- **Environment variables** $p$
  - Dimensions, test piece (design point)
  - Temperature (value of constitutive law or state of thermo-viscoelastic)

- Feature: function of output (example modal frequency)

Simple example: modified Oberst test for 3D weaved composite test
System models: nature & objectives

What is a model
- A function relating input and outputs
- For one or many parametric configurations

Model categories
- Behavior models (meta-models)
  - Test, constitutive laws, Neural networks
  - Difficulties: choice of parametrization, domain of validity
- Knowledge models
  - Physical principles, low level meta-models

Why do we need system models?
Design
- Become predictive: understand, know limitations
- Perform sizing, optimize, deal with robustness
Certify
- Optimize tests: number, conditions
- Understand relation between real conditions and certification
- Account for variability
Maintain during life
- Design full life cycle (plan maintenance)
- Use data for conditional maintenance (SHM)
Equations of motion

- Nominal model (elastic + viscous damping)
  \[
  [M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K(q)]\{q\} = [b(q)]\{u(t)\}
  \]
  \{q\} DOF, \(M\) mass, \(K\) stiffness

- Loads decomposed as spatially unit loads and inputs
  \{F(t)\} = [b] \{u(t)\}

- \{y\} outputs are linearly related to DOFs \{q\} using an observation equation
  \{y(t)\} = [c] \{q(t)\}

- Simple case: extraction \{w_2\}=[0 0 1 0]\{q\}

- More general: intermediate points, reactions, strains, stresses, ...
# Equations of motion

## FEM ↔ Reduction

<table>
<thead>
<tr>
<th></th>
<th>Finite elements</th>
<th>Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Support</strong></td>
<td>Element: line, tria, tetra, ...</td>
<td>FE mesh</td>
</tr>
<tr>
<td><strong>Variable separ.</strong></td>
<td>( w(x, t) = N_i(x)q_i(t) )</td>
<td>( { q(t) } = { T_i } q_i(t) )</td>
</tr>
<tr>
<td><strong>Shape functions</strong></td>
<td>( \varepsilon(x, t) = B_i(x)q_i(t) )</td>
<td>( T_i ) simple FE solutions</td>
</tr>
<tr>
<td><strong>Matrix comp.</strong></td>
<td>( K_{ij} = \int_\Omega B_i^T \Lambda B_j = \sum_g B_i^T (g) \Lambda B_j w_g J_g )</td>
<td>( K_{ijR} = T_i^T K T_j )</td>
</tr>
<tr>
<td><strong>Weak form</strong></td>
<td>Numerical integration</td>
<td>FEM matrix projection</td>
</tr>
<tr>
<td><strong>Assembly</strong></td>
<td>Localization matrix</td>
<td>Boundary continuity, CMS</td>
</tr>
<tr>
<td><strong>Validity</strong></td>
<td>Fine mesh for solution gradients</td>
<td>Good basis for considered loading</td>
</tr>
</tbody>
</table>

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Ritz/Galerkin reduction from full

- **Basis building steps**
  - FEM: cinematically admissible subspace, virtual work principle
  - Reduction: 1) learn, 2) generate basis 3) choose DOF
    \[
    \{ q(p, t) \}_{N} \approx [T]_{N \times NR} \{ q_{R}(p, t) \}_{NR}
    \]

- **Virtual work principle / reduction / Ritz-Galerkin**
  - Matrices
    \[
    [M_{R}(p)] = T^{T}M(p)T, \quad [K_{R}(p)] = T^{T}K(p)T
    \]
  - Loads
    \[
    \{ f(p, t) \} = [b_{R}(p)]\{ u(t) \} = [T^{T}b]\{ u \}
    \]
  - Observations
    \[
    \{ y(p, t) \} = [c_{R}(p)]\{ q_{R}(p, t) \} = [cT]\{ q_{R} \}
    \]

- **Solve time/freq (same model form)**
  \[
  [M_{R}]\{ \ddot{q}_{R} \} + [C_{R}][q_{R}] + [K_{R}][q_{R}] = [b]\{ u(t) \}
  \]
  \[
  \{ y(t, p) \} = [c_{R}][q_{R}]
  \]
Continuous/discrete/reduced models (a brief reminder)

Full order model solvers

- Direct frequency resolution
- Direct time integration (implicit/explicit, first/second order, Newmark, … Gaël Chevallier)

Reduced order model + time/frequency resolution

- Basic reduction: modal superposition, static correction, Guyan, Craig-Bampton, …
- Modern vision of reduction: learning phase, basis building, DOF choice
- Substructuring
- Parametric model reduction, error control

When does reduction become useful?
Basic building blocks?
MATLAB Tutorial : direct frequency response issues

- Step 1: assembly, sparse matrices
- Step 2: point load, collocated displacement, factorization strategies
- Step 3: subspace around resonance, phase collinearity, SVD
- Step 4: Rayleigh-Ritz, reduced FRF
Direct frequency response: $Zq = F$ (step 2)

1. Renumbering (fill in reduction, symbolic factorization, METIS, symrcm, …)
2. Numerical factorization $Z = LU$ or $Z = LDL^T$
3. Forward/backward solve $L(D(L^T q)) = F$

Sparse libraries: Umfpack (lu), MA57 (ldl), Pardiso, Mumps, BCS-Lib, Spooles, Taucs, …
Transfers: what subspace is needed?

- **Experimental modal analysis subspace:**
  - **Modes within bandwidth**
  - **Upper residual** (residual flexibility, static correction, state-space D term)
  - **Lower residual** (rigid body inertia, ...)

Nearby modes = poor representation of static
Normal modes of elastic structure

- **Nominal model (elastic + viscous damping)**
  \[
  \left[ M s^2 + C s + K \right] \{q(s)\} = [b]\{u(s)\}
  \{y(s)\} = [c]\{q(s)\}
  \]

- **Conservative eigenvalue problem**
  \[
  -[M] \{\phi_j\} \omega_j^2 + [K]_{N \times N} \{\phi_j\}_{N \times 1} = \{0\}_{N \times 1}
  \]

- **Properties**
  - $M \succ 0$ & $K \succeq 0 \implies \phi$ real
  - Partial solvers exist
Normal modes of elastic structure

- Orthogonality
  \[ [\phi]^T [M] [\phi] = [\mu_j \omega_j^2] \]
  \[ [\phi]^T [K] [\phi] = [\mu_j \omega_j^2] \]

- Scaling conditions

  - Unit mass

  \[ \{\phi_j\}^T [M] \{\phi_j\} = 1 \]

  \[ [c_s]\{\tilde{\phi}_j\} = 1 \quad \mu_j(c_s) = ([c_i]\{\phi_j\})^{-2} \]

- Unit amplitude

- Principal coordinates

\[
\begin{bmatrix} [I] s^2 + [\Gamma] s + [\omega_j^2] \end{bmatrix} \{p(s)\} = [\phi^T b] \{u(s)\}
\]

\[
\{y(s)\} = [c\phi]\{p(s)\}
\]
Modal contributions & static correction

\[ H(\omega) = [c] \left[ -M \omega^2 + K \right]^{-1} [b] \approx \]

\[ \sum_{j=1}^{N_M} \frac{[c] \{\phi_j\} \{\phi_j\}^T [b]}{-\omega^2 + \omega_j^2} + \sum_{j=N_M+1}^{N} \frac{[c] \{\phi_j\} \{\phi_j\}^T [b]}{\omega_j^2} \]
Response is approximated

\[ q(s) = \left[ \phi_1 \ldots \phi_{NM} \right] [K_{Flex}]^{-1} [b] \]

- within subspace containing modes and flexibility

\[ T = \left[ \phi_1 \ldots \phi_{NR} \right] [K_{Flex}]^{-1} [b] \]

- or modes and residual flexibility

\[ T = \left[ \phi_{1:NM} \right] [K_{Flex}]^{-1} \left[ b - M \left( \phi_{1:NM} \right) \phi_{1:NM}^T b \right] \]
For free structure: static load implies deformation in a uniformly accelerating frame

\[
\{q_F\} = [K]_{Flex}^{-1} [b] = \sum_{j=NB+1}^{N} \left[ \frac{\phi_j}{{\omega_j}^2} \phi_j^T b \right]
\]
Unit imposed displacement

**Applied load**: free modes + static correction = McNeal

**Applied displacement**: dynamic & Static/Guyan condensation

\[
\begin{bmatrix}
K_{II} & K_{IC} \\
K_{CI} & K_{CC}
\end{bmatrix} \begin{bmatrix}
<q_I(s)> \\
q_C(s)
\end{bmatrix} + [Ms^2] \{q\} = \begin{bmatrix}
R_I(s) \\
<0>
\end{bmatrix}
\]

No interior load = dynamic condensation

\[
[T(\omega)] = \begin{bmatrix}
I \\
-Z_{CC}(\omega)^{-1}Z_{CI}(\omega)
\end{bmatrix}
\]

Inertia neglected = static/Guyan

\[
[T] = \begin{bmatrix}
I \\
-K_{CC}^{-1}K_{CI}
\end{bmatrix}
\]
Frequency limit -> Craig-Bampton

Inertia neglected: error associated with $M_{cc}q_c$

When $Z_{cc}(s)$ is singular
⇒
Approximation cannot be valid

Fixed interface modes

Craig-Bampton = guyan/static + fixed interface

$$[T] = \begin{bmatrix} I & 0 \\ K_{cc}^{-1}K_{ci} & \phi_{1:NM,c} \end{bmatrix}$$
Learning strategy variants: POD

Traditional: modes + static correction

\[
T = \begin{bmatrix}
\phi(Z_{cc}(\omega_j)) & K_{CC}(s)^{-1}K_{CV}(s)V_{In} \\
0 & V_{In}
\end{bmatrix}
\perp M, K
\]

Snap-shot Ritz basis

\[
T = \left\{ Z_{CC}(s)^{-1}Z_{CV}(s)V_{In} \right\}_{s \in i\omega_{\text{target}}} \perp M, K
\]

3 out of 100 useful modes
Relatively close static correction

Easily captures wide range
Learning strategy: wave/cyclic

1. Learn using wave (Floquet)/cyclic solutions
2. Build basis with left/right compatibility
3. Assemble reduced model

PhD Elodie Arlaud, 2016

PhD Sternshuss 2008
Outline

1. Learning phase
   1. modes & static responses (bandwidth, inputs) : McNeal, Guyan, Craig-Bampton
   2. POD

2. Basis generation DOF selection
   1. SVD (truncation)
   2. Gramm-Schmidt, conjugate-gradient (Lanczos)
   3. Piecewise learning (sparsity, superelements, Component mode synthesis)

3. Model reduction/modal synthesis/Ritz-Galerkin/virtual work principle
   \[ \{q(x, t)\} = [T(x)]\{q_R(t)\} \Rightarrow Z_R(\omega, p) = T^T[Z(\omega, p)]T \]
   Optimize reduced model usage

4. Beyond LTI (parametric, NL, time varying, ...)
   T independent of p
Tuto steps 3-4

- POD learning
- Rayleigh-Ritz / reduced solve
SVD & variants

\[ A = U S V^H \]

- \( \{X\} \) on sphere in input space transformed in \( \{Y\} = [A]\{X\} \) ellipsoid
- Series of rank one contributions

**Mode**
- \( \{\phi\} \) on unit strain energy sphere output is kinetic energy
- Singular value \( \frac{1}{\omega_j^2} = \frac{\phi_j^T M \phi_j}{\phi_j^T K \phi_j} = 1/\text{Rayleigh quotient} \)

AIAA Journal, Balmes, 1996
Optimize reduced model computation

• Spectral decomposition

\[ \{y\} = c_R [M_R s^2 + K_R]^{-1} b_R u = \sum_{j=1}^{NM} \frac{c \phi_j \phi_j^T b u}{s^2 + 2\zeta_j \omega_j s + \omega_j^2} \]

- Cost is \( O(N^3) \times N_F \) or \( O(N^3) + O((NM \times N_a \times N_s) \times N_F) \)
- The second is obviously much lower for not very small \( N_F \)

• Transition matrix (or matrix exponential) : time domain (free response), space domain (Wave Finite Element)

\[ \{x_{n+1}\} = [A]\{x_n\} = \left[ \Theta_R \begin{bmatrix} \lambda_j \end{bmatrix} \Theta_L^T \right] \{x_n\} = \left[ \Theta_R \begin{bmatrix} \lambda_j^{n+1} \end{bmatrix} \Theta_L^T \right] \{x_0\} \]

- The second is obviously much lower for not very small \( n \)
Random fields \textbf{Karhunen-Loeve}:
\begin{itemize}
  \item input-norm I for all DOFs
  \item output norm spatial correlation \\
  \[ C = \exp[-(|x_1 - x_2| + |y_1 - y_2|)] \]
\end{itemize}

PCA \textbf{Principal Component Analysis}
\textbf{POD} based on snapshot-reduction:
\begin{itemize}
  \item input-norm I on snapshot vectors
  \item output norm I
\end{itemize}

\textbf{Junction modes}:
\begin{itemize}
  \item input-norm I for modes or contact stiffness \\
  \item output norm local stiffness
\end{itemize}

\textbf{Non-linear dimensionality reduction (manifold)}:
\begin{itemize}
  \item More complex relation between parameters
\end{itemize}

Vector independence

- SVD
- Krylov/Lanczos (iterations & conditioning, step5)
- Gram Schmidt
- LU

Multi-frontal solvers / AMLS

- Graph partitioning methods ⇒ group DOFs in an elimination tree with separate branches
- Block structure of reduction basis
- Block diagonal stiffness
- Very populated mass coupling

- Multi-frontal eigensolvers introduce some form of interface modes to limit size of mass coupling
Interface reduction / model size / sparsity

- Craig-Bampton often sub-performant because of interfaces

- Unit motion can be redefined: interface modes
  Fourier, analytic polynomials, local eigenvalue
  5000 -> 500 interface DOFs.

- Disjoint internal DOF subsets

Separate requirements for learning shapes & basis building:
- bandwidth, inputs external & parameter truncation, sparsity

\[2^{6} \times 5000 \times \text{Int} = 74 \text{GB}\]

\[5000^{2} = 200 \text{ MB}\]
MATLAB Tutorial: reduction, full operators

• Step 5: Krylov
• Step 6: sparse reduced model

• Step 7: frequency limit CB
• Step 8: an experimental case of SVD
DOF / sensor selection

Solutions depend on **subspace** NOT basis
Choose DOF you like or that make sense

Ex 1: beam shape functions
- Subspace \( a + bx + cx^2 + dx^3 \)
- Observation \( y = \{w_1, \theta_1, w_2, \theta_2\}^T \)
- Condition of unit on observation gives shape functions \( N_i \)

Ex 2: multibody dynamics: use master nodes
needed \([c][T]\) full rank

\[
\{y\} = [c][T]\{q_R\}
\]

\[
\downarrow
\]

\[
\{y\} = [\hat{c}][\hat{T}]\{y\} = [I \ 0] \{q_c\}
\]
Physical & Modal DOF

• Physical domain:

\[ [M]\{\ddot{q}(t)\} + [C]\{\dot{q}(t)\} + [K]\{q(t)\} = \{f(q, \dot{q}, t)\} \]

• Modal domain:

- mass orthogonality condition \(\phi^T M \phi = I\)
- stiffness orthogonality condition \(\phi^T_j K \phi_j = \omega_j^2\)
- Modal equation

\[ [I]\{\dddot{\alpha}(t)\} + [\Gamma]\{\ddot{\alpha}(t)\} + \begin{bmatrix} \omega_j^2 \end{bmatrix}\{\alpha(t)\} = \{f(\alpha, \dot{\alpha}, t)\} \]

• Modal amplitudes \(\{\alpha\} = [\phi^{-1}]\{q\} = [\phi^T M]\{q\}\)

Associated concepts: force appropriation, modal filter

• Modal energies \(e_j = \frac{1}{2} (\dot{\alpha}_j^2 + \omega_j^2 \alpha_j^2]\)

Modal participations in ODS

- Extract shape = SVD around « resonance »
- Obtain modal amplitudes of nominal modes
- Apparent stiffness/damping consistent for various methods
Modal energy computations

- Does the shape change in NL behavior

\[ 2E_{mj}(t) = \omega^2_j \alpha^2_j(t) + \dot{\alpha}^2_j(t) \]

\[ E_p(t), \quad E_k(t) \]

Modal DOF

- Multi-stage cyclic symmetry (SNECMA).
  - Which stage, which diameter, ...
  - Mistuning (which blade)
Dealing with NL/parameters/damping

Reduced model → Sensors

• Coupling: test/FEM, fluid/structure active control, ...
• Local non-linearities: machining, bearings, contact/friction, ...
• Optimization / uncertainty
Viscoelastic constitutive relations

- Stress is a function of strain history
- Complex modulus in Laplace domain

\[ \sigma(s) = E(s, T, \sigma_0) \varepsilon(s) = (E' + iE'')\varepsilon(s) \]

- Dynamic stiffness linear combination of fixed matrices

\[ Z(E_i, s) = Ms^2 + K_e + \sum_j E_i(s, T, \sigma_0) \left[ \frac{K_{vi}(E_0)}{E_0} \right] \]
Residue iteration: viscoelastic material

\[ [Z(E_i(s), s)]\{q\} = \{F\} \] Damped viscoelastic resp. rewritten as

\[ [Z(E_0, s)]\{q\} = \{F\} - \sum_j (E_i(s) - E_0) \left[ \frac{K_{vi}(E_0)}{E_0} \right] \{q\} \]

Tangent linear system, internal NL/parametric loads

Basis contains

- **Modes** to represent nominal resonances
- **Flexibility** to viscoelastic loads associated with nominal modes

\[
T = \begin{bmatrix} \Phi_{1:NM} & K_o^{-1} \left[ \text{Im}(Z-Z_o) \right] \Phi_{1:NM} \end{bmatrix}
\]

- **Modes** static response to parametric load

**Principle of reduction**

(assumptions on excitation space & freq) unchanged
What does **first order** bring?

- Correct energy distribution
- Accuracy on peaks (modal is over-damped up to 100%)

First order shape: $T = [K_0^{-1} \text{Im}(Z-Z_0)] \phi_4$ $\text{orth}$
Parametric loads & reduction

Space/time decomposition of load $[b_{Contact}]_{N \times Ng}\{p(t)\}$
- Know nothing about $\{p(t)\}_{Ng}$ too large

- $\{p(t)\}$ associated with initial modes $=[[c_{NOR}]\phi_{1:NM}]_{N \times NM}\{q_r(t)\}$
  Static correction for pressure load of elastic normal modes
  $T = [\phi(p_0) K^{-1} b_c c_{NOR} \phi(p_0)]_{N \times NM}$

- Multi-model learning $T = [\phi(p_1) \phi(p_2)_{N \times NM}]_\perp$

- Error control (residue iteration)
  
  $R_d = K_0^{-1} \left\{ [M_0 s^2 + K_0] T q_R - [b_{ext}] u_{ext} + \{f_p(T q_R, p)\} \right\}$

PhD A. Bobillot 2002
Bases for parametric studies

- **Multi-model**

- **Other + residue iteration**

- **Example: strong coupling**
  With heavy fluids: modes of structure & fluid give poor coupled prediction

\[
\begin{bmatrix}
M & 0 \\
C^T & K_p
\end{bmatrix} s^2 + \begin{bmatrix}
K(s) & -C \\
0 & F
\end{bmatrix} \{q, p\} = \begin{bmatrix}
F_{ex} \\
0
\end{bmatrix}
\]

\[
\tilde{S} = S [K_0]^{-1} [C] [F] \\
\tilde{F} = F [T]^{-1} [C] [F]^T 
\]

Orthogonalization:

\[
[T] \rightarrow [T^k] \rightarrow R_d^k = K^{-1} R(q(T^k)) 
\]

Orthog [T^k R_d^k]
• Step 1: Load model
• Step 2: Multi-model reduction
• Step 3: Analyze frequency/damping evolution
• Step 4: Analyze MAC, use modal coordinates
Fixed basis: enormous cost reduction

- Windshield joint complex modes at 500 design points for $\frac{1}{2}$ cost of direct solver
- Campbell diagram: 200 rotations speeds for the cost of 4.
- Squeal instabilities as function of pressure: few pressures sufficient for interpolation

<table>
<thead>
<tr>
<th>$\psi,\lambda$</th>
<th>SOL107</th>
<th>2200s</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi,\omega$</td>
<td>SOL103</td>
<td>300s</td>
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<tr>
<td>$\psi,\lambda$ Reduced</td>
<td>First order Error &lt;4%</td>
<td>490s</td>
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<tr>
<td>$\psi,\lambda(500^*T)$</td>
<td>SOL107</td>
<td>~12 days</td>
</tr>
<tr>
<td>$\psi,\lambda(500^*T)$ reduced</td>
<td>First order Error small</td>
<td>~1000s</td>
</tr>
</tbody>
</table>
Fixed basis reanalysis

- Response surface for system matrices
  \[ T^T Z(p)T \approx f(p, T^T M_i T) \]

But

- still dynamic model
- restitution \( \{q\} = [T]\{q_R\} \)
  provides estimates of all internal states

Response surface/meta-model methodologies

- also predict I/O relation
- but no knowledge of internal state

PGD & HBM methodologies: variable separation of higher dim

\[
\{q(t, p)\} = \sum_i \{T_i \text{space}\}\{T_i \text{time}\} q_i(p)
\]
Conclusions: solvers for dynamics

Continuous/discrete/reduced models (a brief reminder)
Full order model solvers
• Direct frequency resolution
• Direct time integration (implicit/explicit, first/second order, Newmark, ... Gaël Chevallier)

Reduced order model + time/frequency resolution
• Basic reduction: modal superposition, static correction, Guyan, Craig-Bampton, ...
• Modern vision of reduction: learning phase, basis building, DOF choice
• Substructuring
• Parametric model reduction, error control