Exploitation du Théorème de l'Energie Potentielle

En utilisant le théorème de l'énergie potentielle, on retrouve l'expression de l'equilibre global du champ de contrainte solution :

$$\int_\Omega \mathrm{Tr}[\underline{\underline{\sigma}}_{ex}\underline{\underline{\varepsilon}}(\underline{U}^\star)]d\Omega = \int_\Omega \underline{f}^d\underline{U}^\star d\Omega + \int_{\partial\Omega_2} \underline{F}^d\underline{U}^\star dS \quad \forall \underline{U}^\star \text{ CA0}$$

Démonstration :
Si $\underline{U}_{ex}$ est la solution du problème alors tout champ $\underline{U}$ Cinématiquement Admissible est tel que :

$$\underline{U} = \underline{U}_{ex} + \lambda\underline{U}^\star \quad \forall \underline{U}^\star \text{ CA0}$$

Le champ $\underline{U}_{ex}$ étant solution exacte, il correspond au minimum de l'énergie portentielle. Ce minimum s'écrit donc :

$$\frac{d}{d\lambda}\big[E_p(\underline{U}_{ex} + \lambda\underline{U}^\star)\big]_{|\lambda=0} = 0 \quad \forall \underline{U}^\star \text{ CA0}$$

En notant $\underline{\underline{\varepsilon}}_{ex} = \underline{\underline{\varepsilon}}(\underline{U}_{ex})$ et $\underline{\underline{\varepsilon}}^\star = \underline{\underline{\varepsilon}}(\underline{U}^\star)$ on a :

$$\begin{aligned}
E_p(\underline{U}_{ex} + \lambda\underline{U}^\star) = & \frac{1}{2}\int_\Omega \mathrm{Tr}[\underline{\underline{\varepsilon}}_{ex}\underline{\underline{\underline{\underline{K}}}}\underline{\underline{\varepsilon}}(\underline{U}^\star)]d\Omega + \frac{1}{2}\lambda^2\int_\Omega \mathrm{Tr}[\underline{\underline{\varepsilon}}^\star\underline{\underline{\underline{\underline{K}}}}\underline{\underline{\varepsilon}}^\star]d\Omega \\
& + \lambda\int_\Omega \mathrm{Tr}[\underline{\underline{\varepsilon}}_{ex}\underline{\underline{\underline{\underline{K}}}}\underline{\underline{\varepsilon}}^\star]d\Omega \\
& - \Big(\int_\Omega \underline{f}^d\underline{U}_{ex}d\Omega + \int_{\partial\Omega_2}\underline{F}^d\underline{U}_{ex}dS\Big) \\
& - \lambda\Big(\int_\Omega \underline{f}^d\underline{U}^\star d\Omega + \int_{\partial\Omega_2}\underline{F}^d\underline{U}^\star dS\Big)
\end{aligned}$$

donc

$$\begin{aligned}
\frac{d}{d\lambda}\big[E_p(\underline{U}_{ex} + \lambda\underline{U}^\star)\big] = & \lambda\int_\Omega \mathrm{Tr}[\underline{\underline{\varepsilon}}^\star\underline{\underline{\underline{\underline{K}}}}\underline{\underline{\varepsilon}}^\star]d\Omega \int_\Omega \mathrm{Tr}[\underline{\underline{\varepsilon}}_{ex}\underline{\underline{\underline{\underline{K}}}}\underline{\underline{\varepsilon}}^\star]d\Omega \\
& - \Big(\int_\Omega \underline{f}^d\underline{U}^\star d\Omega + \int_{\partial\Omega_2}\underline{F}^d\underline{U}^\star dS\Big)
\end{aligned}$$

donc

$$\frac{d}{d\lambda}\big[E_p(\underline{U}_{ex} + \lambda\underline{U}^\star)\big]_{|\lambda=0} = \int_\Omega \mathrm{Tr}[\underline{\underline{\sigma}}_{ex}\underline{\underline{\varepsilon}}^\star]d\Omega - \int_\Omega \underline{f}^d\underline{U}^\star d\Omega - \int_{\partial\Omega_2}\underline{F}^d\underline{U}^\star dS = 0$$

CQFD