Préparation DUT 1

mercredi 2 septembre 2020 09:09

Objectifs :

- Ne pas avoir de redoublant de DUT à cause des maths de la mécanique !
- Des rappels, mais seulement ce qui va être utile, et pas sous forme de cours de maths
- Bien identifier les points à travailler en autonomie par la suite
- Avoir des trucs pour aller plus vite dans les calculs / gommer l'écart d'efficacité avec les prépas

$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $B = \begin{pmatrix} -1 & 1 & -1 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 6 & -1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$

$$P_1 = \underbrace{AB}C = \begin{pmatrix} -1 & 1 & -1 \\ -2 & 2 & -2 \\ -3 & 3 & -3 \end{pmatrix} \begin{pmatrix} 1 & 6 & -1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -6 & 0 \\ 0 & -12 & 0 \\ 0 & -18 & 0 \end{pmatrix}$$

AB C

$$BC = \begin{pmatrix} -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 6 & -1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -6 & 0 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0 & -6 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -12$$

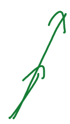
Vecteurs :

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} \wedge \begin{pmatrix} D \\ E \\ F \end{pmatrix} = \begin{pmatrix} BF - CE \\ -AF + CD \\ AE - BD \end{pmatrix}$$

A D

B E

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

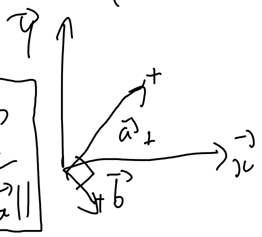


$$\det \begin{pmatrix} 3 & 1 \\ 3 & -1 \end{pmatrix} = 3 \times -1 - 3 \times 1 = -6$$

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \Rightarrow$$ les vecteurs sont orthogonaux

pas orthonormée $\|\vec{b}\| = \sqrt{1+1} \neq 1$

$$\boxed{\vec{a}_n = \frac{\vec{a}}{\|\vec{a}\|}}$$



$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{a}_n = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

$$\vec{b}_n = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$\vec{V}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \text{ et } \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} = \vec{V}_2$$

$V_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \end{pmatrix}$, et $\begin{pmatrix} -1/\sqrt{2} \\ 0 \end{pmatrix} = V_2$

$$\vec{V_3} = \vec{V_1} \wedge \vec{V_2} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$\vec{V_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\vec{V_2} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ et $\vec{V_3} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$|A| = det(A)$

$1+1$

$mL + mC$

$$\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 1 \times (-1)^{1+1} \times \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 1 \times (-1-1)$$

$$+ -1 \, (-1)^{1+2} \times \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} \qquad + -1 \times -1 \times (1-1)$$

$$+ 1 \, (-1)^{1+3} \times \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} \qquad + 1 \times (-1-1)$$

$$= -4$$

barre

colonne N° 2 + colonne N°1

$(2+1)$ $(2+3)$

$$\begin{vmatrix} 1 & 0 & 0 \\ -1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} = 1 \times -1^{(1+1)} \times -4 = -4$$

$$\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} =$$

$$\begin{matrix} 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 \end{matrix}$$

← produit diagonales

addition

-1 $\qquad -3$ $\qquad \Rightarrow$ résultat = droite − gauche

$= -3 - 1$

système :

$$\alpha \vec{w_1} + \beta \vec{w_2} + \gamma \vec{w_3} = \vec{0}$$

s'il y a une solution ⟹ vecteurs dépendants

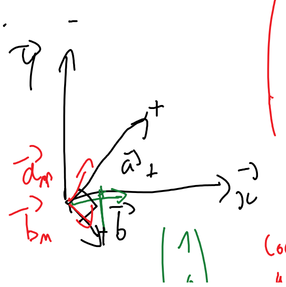
$$\begin{cases} \alpha + 0 - \gamma = 0 \\ 0 + \beta - \gamma = 0 \\ -\alpha + \beta + 0 = 0 \\ \quad -\beta + \gamma = 0 \end{cases}$$

$\alpha = \gamma$

$\beta = \gamma$ $\qquad \alpha = \beta = \gamma$ solution

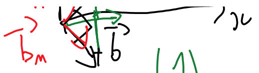
OK

OK



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{array}{cc} \vec{a_n} & \vec{b_n} \\ \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \end{array} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Coord ancienne $\qquad$ P $\qquad$ Coord nvel b...



$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Coord ancienne base

P $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{pmatrix}$ Coord nvel base

$$\vec{V} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = P\ \vec{V}'$$

$$P \to \left(\vec{V}_1\ \vec{V}_2\ \vec{V}_3 \right) \leftarrow$$ vecteurs de la nouvelle base exprimés dans l'ancienne base

$$\vec{V}' = P^{-1}\vec{V}$$

$$P^{-1} = \frac{1}{\det(P)}\,\mathrm{com}(P)^T = -\frac{1}{4}\,\mathrm{com}(P)^T$$

$$P = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & \cancel{2} \\ 3 & 4 \end{pmatrix}$$

$$\mathrm{com}(Q) = \begin{pmatrix} 4 \times (-1)^{1+1} & -3 \\ -2 & 1 \end{pmatrix}$$

$$\mathrm{com}(Q)^T = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

$$\mathrm{com}(P) = \begin{pmatrix} (-1)^{1+1}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & \dots \\ \vdots & \end{pmatrix}$$

$$\mathrm{com}(P) = \begin{pmatrix} -2 & 0 & -2 \\ 0 & -2 & -2 \\ -2 & -2 & 0 \end{pmatrix} = \mathrm{com}(P)^T \quad (\text{car symétrique})$$

$$\vec{V}' = -\frac{1}{4}\begin{pmatrix} -2 & 6 & -2 \\ 0 & -2 & -2 \\ -2 & -2 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -6+4 \\ -2+4 \\ -6-2 \end{pmatrix} \times -\frac{1}{4} = \begin{pmatrix} 1/2 \\ -1/2 \\ 2 \end{pmatrix}$$

b) $\vec{V}' = P^{-1}\vec{V}$

$$A' = P^{-1} A P$$

$\uparrow$ matrice 3×3

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$

$$M' = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{3}{2} & -\frac{1}{2} \\ 1/2 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 3 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$(2\ \ 1\ \ 1)$ $|\quad\ \ +|\quad |2-\lambda\quad 1\quad 1|$

$$A_1 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix} \qquad |A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix}$$

col. 1 + col. 2

$$= \begin{vmatrix} 3-\lambda & 1 & 1 \\ 3-\lambda & 2-\lambda & 1 \\ 2 & 1 & 4-\lambda \end{vmatrix}$$

ligne 1 - ligne 2

$$= \begin{vmatrix} 0 & -1+\lambda & 0 \\ 3-\lambda & 2-\lambda & 1 \\ 2 & 1 & 4-\lambda \end{vmatrix}$$

$$= (\lambda-1)(-1)^{1+2}\left((3-\lambda)(4-\lambda) - 2\right)$$

$$= (1-\lambda)(\lambda^2 - 7\lambda + 10)$$

$\Delta = 49 - 4\times 10 = 9$

Valeurs propres

$$\lambda_3 = 1 \qquad \lambda_1 = \frac{+7-\sqrt{9}}{2} = 2 \qquad \lambda_2 = \frac{7+\sqrt{9}}{2} = 5$$

$$= (1-\lambda)(\lambda-2)(\lambda-5)$$

$$\Rightarrow A_1' = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Diagonalisable car 3 valeurs propres distinctes (A1 déjà diagonalisable car symétrique)

Vecteurs propres : $A\vec{V}_i = \lambda_i \vec{V}_i \qquad i = 1, 2, 3$

$$A_1 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

$$(A - \lambda_i I)\vec{V}_i = \vec{0}$$

$\lambda_i = 1$

$$\begin{pmatrix} 2-1 & 1 & 1 \\ 1 & 2-1 & 1 \\ 1 & 1 & 4-1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \vec{0}$$

$$\Leftrightarrow \begin{cases} \alpha + \beta + \gamma = 0 \\ \alpha + \beta + \gamma = 0 \\ \alpha + \beta + 3\gamma = 0 \end{cases} \Rightarrow \begin{matrix} \gamma = 0 \\ \alpha = -\beta \end{matrix} \Rightarrow \vec{V}_1 \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

$\underline{\lambda_2 = 2}$

$$\begin{cases} y + z = 0 \\ x + z = 0 \\ x + y + 2z = 0 \end{cases} \Leftrightarrow \begin{cases} x + z = 0 \\ y + z = 0 \end{cases}$$

$$\vec{u}_2 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$\underline{\lambda_3 = 5}$

$$\begin{cases} -3x + y + z = 0 \\ x - 3y + z = 0 \\ x + y - z = 0 \end{cases} \Leftrightarrow \begin{cases} x - y = 0 \\ x + y - z = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = y \\ 2x = z \end{cases}$$

$$\vec{u}_3 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$\lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 5$

$$A_1' = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow P_1 = \Big(\underbrace{\vec{V_3}\ \vec{V_2}\ \vec{V_1}}_{}\Big) = \begin{pmatrix} 1 & 1 & 1/\sqrt{2} \\ 1 & 1 & -1/\sqrt{2} \\ 2 & -1 & 0 \end{pmatrix}$$

base des vecteurs propres

$A_1' = P_1^{-1} A_1 P_1$ à vérifier

$$A_2 = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} \Rightarrow \det(A_2 - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & 0 & 2-\lambda \end{vmatrix}$$

som des 3 lignes $\rightarrow$

$$= \begin{vmatrix} 2-\lambda & 2-\lambda & 2-\lambda \\ -1 & 1-\lambda & -1 \\ 1 & 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & 0 & 2-\lambda \end{vmatrix}$$

col 1 - col 2   col. 3 - 2

$$= (2-\lambda) \begin{vmatrix} 0 & 1 & 0 \\ -1-1+\lambda & 1-\lambda & -1-1+\lambda \\ 1 & 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\, 1\, (-1)^3 \begin{vmatrix} \lambda-2 & \lambda-2 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)(\lambda-2)(-1) \begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2 (2-\lambda-1)$$

$$= \underbrace{(2-\lambda)^2}_{} (1-\lambda)$$

racine double $\Rightarrow$ diagonalisable ssi il existe 2 vecteurs propres indépendants

$(A_2 - \lambda I)\vec{V} = \vec{0}$

ici, $\lambda = 2$

$$\Leftrightarrow \begin{pmatrix} 2-2 & 1 & 1 \\ -1 & 1-2 & -1 \\ 1 & 0 & 2-2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0}$$

$$\begin{cases} y + z = 0 \\ -x - y - z = 0 \\ x = 0 \end{cases} \Rightarrow y = -z \Rightarrow \vec{V} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

1 seul vecteur propre indépendant
$\Rightarrow A_2$ pas diagonalisable

Gradient, divergence, rotationnel

a) On pose P(x,y,z)=$x^2+\frac{y}{zx}-z^3$. Calculer dP et $\overrightarrow{grad}$(P)

b) Calculer div($\overrightarrow{V}$)et $\overrightarrow{rot}(\overrightarrow{V})$, où $\overrightarrow{V}\begin{pmatrix} xy-zy \\ \frac{z}{xy} \\ \frac{x^2y}{z} \end{pmatrix}$

a) $\frac{\partial P}{\partial x} = 2x - \frac{y}{zx^2}$

$\frac{\partial P}{\partial y} = \frac{1}{zx}$

$\frac{\partial P}{\partial z} = -\frac{y}{z^2x} - 3z^2$

$\Rightarrow \overrightarrow{grad}(P) = \begin{pmatrix} \frac{\partial P}{\partial x} \\ \frac{\partial P}{\partial y} \\ \frac{\partial P}{\partial z} \end{pmatrix}$

$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz$

$dP = \overrightarrow{grad}(P) . \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$

b) $div(\overrightarrow{V}) = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} . \begin{pmatrix} xy-zy \\ z/xy \\ x^2y/z \end{pmatrix} = y - \frac{z}{xy^2} - \frac{x^2y}{z^2}$

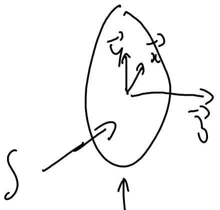
$\overrightarrow{rot}(\overrightarrow{V}) \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \wedge \begin{pmatrix} xy-zy \\ z/xy \\ x^2y/z \end{pmatrix} = \begin{pmatrix} \frac{x^2}{z} - \frac{1}{xy} \\ -y - 2\frac{xy}{z} \\ -\frac{z}{x^2y} + z - x \end{pmatrix}$

$\partial/\partial x \quad xy - zy$

$\partial/\partial y \quad z/xy$

Stokes

Soit $\overrightarrow{V}\begin{pmatrix} e^x+2y+\sin(x^2z) \\ 12x+\arctan(yz) \\ \cos(xyz) \end{pmatrix}$, et C la courbe définie par $x^2+y^2=R^2$ parcourue dans le sens trigonométrique.
Calculer la circulation de $\overrightarrow{V}$ le long de C.



$\overrightarrow{dS} = \overrightarrow{n}\, dS = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dx\, dy$

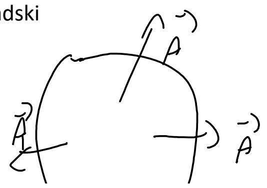
$\hat{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\oint \overrightarrow{V}.\overrightarrow{dl} = \iint_S \overrightarrow{rot}(\overrightarrow{V}) . \overrightarrow{dS} = \iint \overrightarrow{rot}(\overrightarrow{V}) . \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dx\, dy$

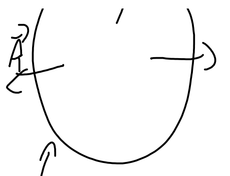
$\Rightarrow \overrightarrow{rot}(\overrightarrow{V}) . \overrightarrow{z} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} \wedge \begin{pmatrix} V_x \\ V_y \end{pmatrix} = 12 - 2 = 10$

$\oint \overrightarrow{V}.\overrightarrow{dl} = \iint_S 10\, dx\, dy = 10 \underbrace{\iint_S dx\, dy}_{S} = 10\, S = 10\pi R^2$

Ostrogradski



$\oiint_S \overrightarrow{A}.\overrightarrow{dS} = \iiint_V div(\overrightarrow{A})\, dV$



$$\oiint_S \vec{A}\cdot dS = \iiint_V div(\vec{A})\, dV$$

sphère surface S de sa "peau"

$$\vec{A}\begin{pmatrix} 5x+\sin(y^2 z)\\ atan(xz)+4y\\ \cos(xy)-6z\end{pmatrix} \qquad dV = dx\,dy\,dz$$

$$div(\vec{A}) = \begin{pmatrix} \partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{pmatrix}\cdot\vec{A} = 5+4-6 = 3$$

$$\iiint_V div(\vec{A})\, dV = \iiint_V 3\, dV = 3\underbrace{V}_{\text{volume de la sphère}} = 4\pi R^3$$

⚠ l'opérateur $\vec{\nabla}$ ne fonctionne que pour les coordonnées cartésiennes

$$\vec{OM} = x\,\vec{e_x} + y\,\vec{e_y} + z\,\vec{e_z}$$

$$\vec{OM} = r\,\vec{e_r} \qquad \vec{e_r} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

$$d\vec{OM} = dr\,\vec{e_r} + r\,\underbrace{d\vec{e_r}}_{d\theta\,\vec{e_\theta}}$$

$$U \qquad dU = \frac{\partial U}{\partial r}dr + \frac{\partial U}{\partial \theta}d\theta + \frac{\partial U}{\partial z}dz$$

$$\vec{grad}(U) = A\,\vec{e_r} + B\,\vec{e_\theta} + C\,\vec{e_z}$$

$$A = \frac{\partial U}{\partial r} \qquad B = \frac{1}{r}\frac{\partial U}{\partial \theta} \qquad C = \frac{\partial U}{\partial z}$$

$$dP = \vec{grad}(P)\cdot\begin{pmatrix}dx\\ dy\\ dz\end{pmatrix}$$

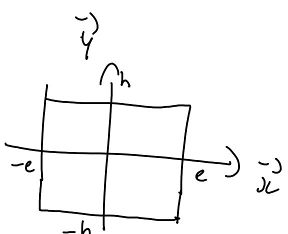
$$dU = \vec{grad}(U)\begin{pmatrix}dr\\ d\theta\\ dz\end{pmatrix}$$

⚠ voir formulaire pour gradient div, rot en cylindrique/sphérique

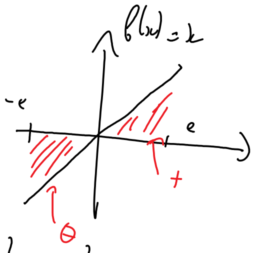
Calcul d'intégrales de surface

a) $\int_{-h}^{h}\int_{-e}^{e} xy^2\, dx\, dy$

b) $\int_{-h}^{h}\int_{-e}^{e} x^2y^2\, dx\, dy$



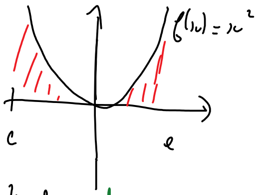
a) $$\int_{-e}^{e} x y^2\, dx = y^2 \underbrace{\int_{-e}^{e} x\, dx}_{=0} \longrightarrow$$



$$\left[\frac{x^2}{2}\right]_{-e}^{e} = \frac{e^2}{2} - \frac{(-e)^2}{2} = 0$$

$\Rightarrow$ fonction impaire comme x, x^3, $\sin(x)$, $\sin^3(x)$

b) $\displaystyle\int_{-h}^{h}\int_{-e}^{e} x^2 y^2 \, dx\, dy$

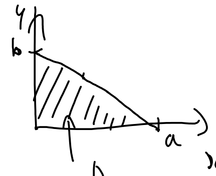


$$\int_{-h}^{h}\int_{-e}^{e} x^2 y^2 \, dx\, dy = \int_{-h}^{h} y^2 \int_0^e 2x^2 \, dx \, dy$$

$$= \int_0^h 2y^2 \times 2 \left[\frac{x^3}{3}\right]_0^e dy$$

$$= 4 \, \frac{e^3}{3} \, \frac{h^3}{3}$$

c)



$y \leq -\frac{b}{a}x + b$

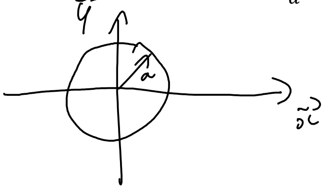
$$\iint_D x^2 \, dx\, dy = \int_0^a \int_0^{-\frac{b}{a}x+b} x^2 \, dy\, dx$$

$$= \int_0^a \left(-\frac{b}{a}x + b\right) x^2 \, dx$$

$$= -\frac{b}{a}\,\frac{a^4}{4} + b\,\frac{a^3}{3}$$

$$= b\,a^3/12$$

$\iint_D (x^3 + y^3)\,dS$, où D ={(x,y)∈(R⁺)², $\frac{x^2}{a^2} + \frac{y^2}{a^2} \leq 1$}



$\Rightarrow$ Coordonnées polaires :
changement de variable : $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$,

$$dS = dx\,dy \rightarrow r\,dr\,d\theta$$

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} \end{bmatrix}$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

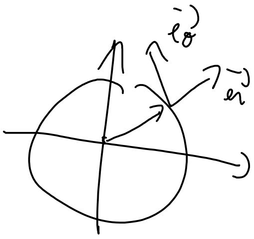
$$\iint_{S(x,y)} f(x,y)\,dx\,dy = \iint_{S(r,\theta)} f(x(r,\theta), y(r,\theta)) \det(J)\,dr\,d\theta$$

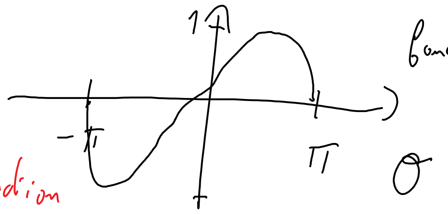
$$\det(J) = r\cos^2\theta + r\sin^2\theta = r$$

$$\int_{-\pi}^{\pi}\int_0^{a}\left(r^3\cos^3\theta + \cancel{r^3\sin^3\theta}\right) r\,dr\,d\theta$$

$\rightarrow 0$ car fonction impaire en θ entre bornes symétriques

$\sin(\theta)$ et $\sin^3(\theta)$ fonctions impaires





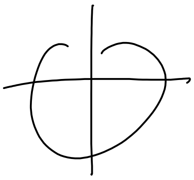
$$\cos^3\theta = \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^3 = \frac{1}{8}\left(e^{3i\theta} + 3e^{2i\theta}e^{-i\theta} + 3e^{i\theta}e^{-2i\theta} + e^{-3i\theta}\right)$$

$$= \frac{1}{8}\left(\left(e^{3i\theta}+e^{-3i\theta}\right) + 3\left(e^{i\theta}+e^{-i\theta}\right)\right)$$

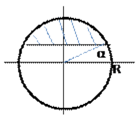
$$= \frac{1}{4}\left(\cos(3\theta) + 3\cos\theta\right)$$

$$\int_{-\pi}^{\pi}\int_0^{a} r^4 \frac{1}{4}\left(\cos 3\theta + 3\cos\theta\right) dr\,d\theta$$

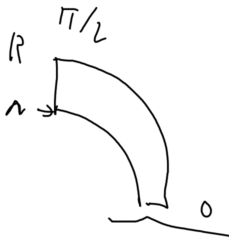
$$\frac{1}{4}\frac{a^5}{5}\left[\frac{+\sin 3\theta}{3} + 3\sin\theta\right]_{-\pi}^{\pi} = \frac{1}{4}\frac{a^5}{5}\left(0-0\right) = 0$$



Exercice à faire : la même intégrale mais sur domaine elliptique
$\frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1$



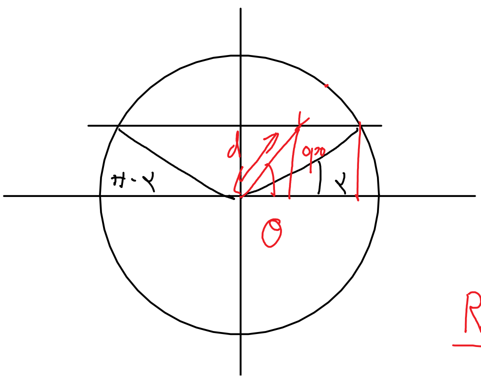
$\iint_D y\,dS$, où D est le domaine hachuré :



$\rightarrow$ On passe en coordonnées polaires

bornes : $\alpha \leq \theta \leq \pi - \alpha$

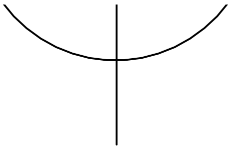
$$\boxed{\sin\alpha = \frac{opp}{R}}$$



$$\sin\theta = \frac{opp}{d}$$

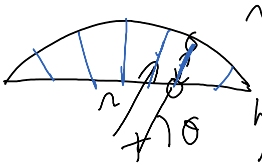
$$d = \frac{opp}{\sin\theta} = \frac{R\sin\alpha}{\sin\theta}$$

$$\frac{R\sin\alpha}{\sin\theta} \leq r \leq R$$

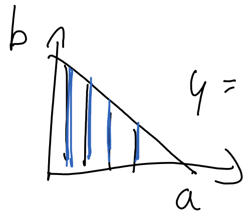


$$\frac{R\sin\alpha}{\sin\theta} \leq r \leq R$$

$r \leq R$ $\qquad r_{min} \leq r \leq r_{maxi}$



$h \in R\sin$ $\qquad \sin\theta = \frac{R\sin\alpha}{h}$



$$y = -\frac{b}{a}x + b \;\Big|\; 0 \leq y \leq -\frac{b}{a}x + b$$

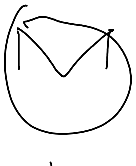
$$\iint_D y \, dS = \int_\alpha^{\pi-\alpha} \int_{\frac{R\sin\alpha}{\sin\theta}}^{R} r\sin\theta \; r\,dr\,d\theta$$

$$= \int_\alpha^{\pi-\alpha} \sin\theta \left[\frac{r^3}{3}\right]_{\frac{R\sin\alpha}{\sin\theta}}^{R} d\theta$$

$$= \int_\alpha^{\pi-\alpha} \frac{\sin\theta}{3}\left(R^3 - \frac{R^3\sin^3\alpha}{\sin^3\theta}\right) d\theta$$

$$= \frac{R^3}{3}\int_\alpha^{\pi-\alpha}\left(\sin\theta - \frac{\sin^3\alpha}{\sin^2\theta}\right) d\theta$$

$$= \frac{R^3}{3}\left[-\cos\theta + \sin^3\alpha \,\mathrm{cotan}\,\theta\right]_\alpha^{\pi-\alpha}$$



$\cos\alpha = -\cos(\pi-\alpha)$

$\sin\alpha = \sin(\pi-\alpha)$

$$= \frac{R^3}{3}\left(\underbrace{-\cos(\pi-\alpha)+\cos\alpha}_{2\cos\alpha} + \sin^3\alpha\left(\frac{\cos(\pi-\alpha)}{\sin(\pi-\alpha)} - \frac{\cos\alpha}{\sin\alpha}\right)\right)$$

$$= \frac{R^3}{3}\left(2\cos\alpha + \sin^3\alpha\left(\frac{-2\cos\alpha}{\sin\alpha}\right)\right)$$

$$= 2\cos\alpha\,\frac{R^3}{3}\left(1 - \sin^2\alpha\right)$$

$$= \frac{2}{3} R^3 \cos^3 \alpha$$

a) $y'' - 5y' + 6y = xe^x$

① Solution homogène de $y'' - 5y' + 6y = 0$

$r^2 - 5r + 6 = 0 \Rightarrow \Delta = 25 - 4 \times 6 = 1$

$\Rightarrow r_1 = \frac{+5-1}{2} = 2$

$r_2 = \frac{5+1}{2} = 3$

S.h. : $y(x) = C_1 e^{2x} + C_2 e^{3x}$

Condition aux limites

ex : $y(0) = 0$

$y'(0) = 0$

② Sol. part

$$\begin{cases} C_1'(x) e^{2x} + C_2'(x) e^{3x} = 0 & \times 2 \quad \times 3 \\ C_1'(x)\, 2e^{2x} + C_2'(x)\, 3e^{3x} = xe^{x} & \times -1 \quad \times -1 \end{cases}$$

$$\begin{cases} 2C_2'(x) e^{3x} - C_2'(x)\, 3e^{3x} = -xe^{x} \\ 3C_1'(x) e^{2x} - C_1'(x)\, 2e^{2x} = -xe^{x} \end{cases}$$

$$\begin{cases} C_2'(x) = xe^{x}e^{-3x} = xe^{-2x} \\ C_1'(x) = -xe^{-x} \end{cases}$$

$(u \times v)' = u'v + uv'$

$uv' = (uv)' - u'v$

$\int -xe^{-x}\,dx$

$u = x \qquad u' = 1$

$v' = -e^{-x} \qquad v = e^{-x}$

$C_1(x) = \int -xe^{-x}\,dx = xe^{-x} - \int e^{-x}\,dx$

$C_1(x) = xe^{-x} + e^{-x} = e^{-x}(x+1)$

$$C_2(x) = -\left(\frac{x}{2}+\frac{1}{4}\right)e^{-2x}$$

$$\Rightarrow (x+1)e^{x} - \left(\frac{x}{2}+\frac{1}{4}\right)e^{x} = \left(\frac{x}{2}+\frac{3}{4}\right)e^{x}$$

Solution générale : $y(x)=C_1 e^{2x} + C_2 e^{3x} + \left(\frac{x}{2}+\frac{3}{4}\right)e^{x}$.

c) $x^2 y''(x) - x\, y'(x) + y(x) = Ln(x)$ en effectuant le changement de variable adéquat $x=e^t$

$$y''(x) = \frac{d^2y}{dx}$$

$$dx = e^t\,dt \Rightarrow \frac{dt}{dx} = \frac{1}{e^t} = e^{-t}$$

- $y'(x) = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = y'(t)\,e^{-t}$
- $y''(x) = \frac{d}{dx}\left(y'(t)\,e^{-t}\right) = \underbrace{\frac{dy'(t)}{dx}}_{\frac{dy'(t)}{dt}\frac{dt}{dx}} e^{-t} + y'(t)\underbrace{\frac{de^{-t}}{dx}}_{\frac{de^{-t}}{dt}\frac{dt}{dx}}$

$$= y''(t)\,e^{-2t} + y'(t)\left(-e^{-2t}\right) = e^{-2t}\left(y''(t) - y'(t)\right)$$

$$\underbrace{e^{2t}\,e^{-2t}}_{e^0}\left(y''(t)-y'(t)\right) - e^{t}y'(t)e^{-t} + y(t) = \ln(e^t) = t$$

$y''(t) - 2y'(t) + y(t) = t$ ⇒ c'est l'équation du b)

$$r^2 - 2r + 1 = 0 = (r-1)^2$$

⇒ Solution homogène de type $(C_1 \cancel{x}\,t + C_2)\,e^{\cancel{x}\,t} = y(\cancel{x}\,t)$

⇒ Solution particulière ? ↗ variation de la constante
↘ "au pif" : $a\cancel{x}\,t + b$, on essaye !

$y''(\cancel{x}\,t) = 0$ $\qquad y'' - 2y' + y = 0 - 2a + a\cancel{x}\,t + b$

$y''(x) = 0$

$y'(x) = a$

$$y'' - 2y' + y = 0 - 2a + ax + b$$
$$= ax + b - 2a \overset{?}{=} x$$

$a = 1$
$b = 2$ $\Rightarrow$ ça marche ! c'est une sol. particulière

Sol. générale : $y(x) = (C_1 x + C_2) e^{x} + x + 2$

$x = e^t$ donc $t = \ln x$: $y(x) = ((C_1 \ln x + C_2)\, x + \ln(x) + 2$